Karlsruhe Institute of Technology

Interaction and disorder in edge states of two-dimensional topological insulators

Wechselwirkung und Unordnung in Randzuständen zweidimensionaler topologischer Isolatoren

Diploma Thesis of

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Erklärung zur Selbständigkeit

Hiermit erkläre ich, die vorliegende Arbeit selbstständig angefertigt und dabei nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

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1. Introduction

Topology, disorder and strong correlations in low dimensional systems are paradigmatic concepts in modern condensed matter physics. A central example of materials in which the combination of these concepts plays a fundamental role are two dimensional $\mathbb{Z}_2$ topological insulators [1, 2, 3, 4, 5]. $\mathbb{Z}_2$ topological insulators (TIs) are novel phases of quantum matter which cannot be adiabatically connected to conventional insulators and semiconductors. They are characterized by an insulating gap in the bulk and a pair of mutually time reversed gapless edge or surface states. These edge states differ fundamentally from conventional one-dimensional quantum liquids described by the Tomonaga-Luttinger liquid (LL) in the sense that they are topologically protected against elastic backscattering due to the combination of strong spin-orbit coupling (SOC) and conserved time reversal invariance (TRI). Furthermore the quasiparticles' spin and momentum are locked due to the spin-orbit interaction. Because of this well-defined projection of spin onto the momentum direction the one-dimensional edge state was termed helical Luttinger liquid (HLL).

This new class of materials has been experimentally realized in HgTe/CdTe quantum wells [5], which exhibit two distinct topological phases, both having a gap in the bulk spectrum but differing by edge properties. While the topologically trivial insulator has no edge states, the edge of the topologically nontrivial insulator is described by gapless helical fermions. Such a system shows the quantum spin Hall effect first predicted in the context of graphene with a spin-orbit coupling term [3]. The phase transition between the two topologically nonequivalent phases is driven by increasing the thickness of the HgTe quantum well beyond some critical value, which leads to a band gap inversion [4]. In this situation the edge channels' conductance is close to twice the conductance quantum $G = 2e^2/h$ as expected for ballistic transport. However, this universal value is only approached in very short samples (less than 1 μm long). Conductance of longer samples is in fact lower [6], indicating electron backscattering or possibly, localisation effects. Mechanisms of electron backscattering are a matter of ongoing debate. Possible sources - two-particle backscattering and Kondo impurities - have been discussed in Refs. [7, 8, 9, 10]. Recently, it has also been proposed that inelastic scattering processes could lead to deviations from universal conductance [11, 12]. However, all of these studies have been constricted to short edge channels and long samples have yet to be studied theoretically.

Aside from 2D $\mathbb{Z}_2$ TI, other classes of materials can have topological character too. First, 3D counterparts of the $\mathbb{Z}_2$ TI have been experimentally observed in samples of $Bi_{1-x}Sb_x$ compounds e.g. in Ref. [13]. In this case the surface constitutes a two-dimensional topologically protected metal that emerges due to the inversion of the 3D bulk gap [14]. Not

only insulators but also superconductors may have topological character. In this case the system has a superconducting gap in the bulk but gapless excitation on the surface. In the case of spin-triplet superconductors (symmetry classes D and DIII) these excitations have a character of Majorana fermions [15]. These systems have received a striking amount of attention in the context of topological quantum computation.

In this thesis we study how the interplay of interactions and disorder affects the transport properties of long samples of 2D TI. Apart from the relevance to fundamental physics, the study of interaction and disorder in edge states of 2D TI is also of prime importance for possible technological applications e.g. in the field of spintronics. To this end we derive a microscopic model describing helical fermions with broken S_z symmetry subject to weak disorder and interactions. We then proceed to calculate the AC and DC conductivity of this system using a kinetic equation approach in different regimes of temperature. This enables us to make statements about the robustness of topological protection and the relevance of different scattering mechanisms. Furthermore, we include Luttinger liquid effects that naturally arise due to the low dimensionality of the system under consideration.

1.1. Outline

This work is organized as follows: In the first part, Ch. 2-4, we lay the groundwork for this thesis.

Chapter 2 is devoted to topological insulators. First we give an overview of the general role of topology in condensed matter physics in Sec. 2.1 before introducing the physical system under consideration, $\mathbb{Z}_2$ topological insulators, in Sec. 2.2. There we find that the gapless edge modes at the system's boundary can be described by one-dimensional helical fermions.

This gives the motivation to remind ourselves of some basics of one-dimensional physics in Ch. 3. There we discuss effects of interaction (Sec. 3.5) and disorder (Sec. 3.6) in low dimensional systems as well as important theoretical tools (Sec. 3.1, Sec. 3.4) and models (Sec. 3.2) to describe them.

Lastly, in Ch. 4 we introduce the Keldysh formalism for treating nonequilibrium systems. We start by deriving the kinetic equation for fermions from the more fundamental nonequilibrium theory in Sec. 4.3 and then apply the developed formalism to the model of helical fermions in Sec. 4.4. The results motivate the treatment of helical edge states by a kinetic equation which will be carried out in the main part.

The main goal of this work is to study transport properties of helical edge states in the presence of disorder and interaction. This goal is achieved in three steps. First we calculate the AC conductivity of helical fermions in a long wire in Ch. 5 by using a kinetic equation approach. This analysis is perturbative and gives us insight into the main scattering mechanisms that determine the transport properties of the system. We complement this treatment by deriving an effective Hamiltonian, encapsulating the most relevant scattering terms, by using methods of conformal field theory. In Ch. 6 we use a combination of exact analytical manipulations and numerical treatment to solve the integro-differential equations obtained from the kinetic equation. This enables us to make predictions about the DC conductivity of the edge modes. Finally, we include Luttinger liquid effects into our treatment in Ch. 7 by using simple scaling arguments in combination with bosonization.

1.2. Definitions and conventions

Troughout the thesis the following conventions are used:

- Unless stated otherwise, we use units where $\hbar = k_B = v_F = c = 1$.
- Second quantized operators will be denoted with a hat to distinguish them from quantum fields or other objects. An exception is Ch. 4, where a hat denotes matrix structure in Keldysh space.
- The Hamiltonian will be defined as "grand canonical", i.e. the chemical potential is already included in the definition of the Hamiltonian as $\hat{H} = \hat{H}' - \mu\hat{N}$. Here $\hat{H}'$ is the "conventional" Hamiltonian.
- The ratio of Fermi energy and temperature is denoted by $\zeta = k_F/T$. The quantum number for chirality is generally denoted by η. If η appears as a multiplicative factor we choose the convention $\eta = +1$ for right movers and $\eta = -1$ for left movers.
- The Fermi-Dirac distribution is given by $f^0_{\eta,k} = (1 + \exp\{(\epsilon_{\eta,k} - \mu)/T\})^{-1}$, where $\epsilon_{\eta,k} = \eta k$ is the dispersion relation of helical fermions. If we measure momentum in units of temperature we choose the variables x,y,z and the Fermi distribution is denoted by $n(x) = (1 + e^x)^{-1}$. Note that x,y are also used as real space coordinates. However, we believe it is always clear from the context what is meant.
- We adopt the bosonization conventions as well as the notation for bosonic fields from Sénéchal [16]. Notation conventions in the chapter about non-equilibrium field theory are adopted from Kamenev [17]
- We choose the following representation of Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Integrals without specified limits represent integration over the whole real axis: $\int \mathrm{d}x \equiv \int_{-\infty}^{\infty} \mathrm{d}x$
- Commonly used abbreviations:
  - TI: topological insulator.
  - SOC: spin-orbit coupling, TRI: time reversal invariance.
  - QHE: quantum Hall effect, QSHE: quantum spin Hall effect.
  - LL: Luttinger liquid, HLL: helical Luttinger liquid.
  - IP: integration by parts, OPE: operator product expansion.
  - RG: renormalization group, CFT: conformal field theory.
  - SIA: structural inversion asymmetry, BIA: bulk inversion asymmetry.

2. Topological insulators

In this chapter we give a short introduction into topological insulators in condensed matter physics and derive the effective model for the gapless states that emerge at their edges. For a further study of the field we refer the reader to one of the many good reviews on the topic of topological materials [1, 2, 18, 19, 20].

Let us begin by defining what we mean by topological insulator. A topological insulator (superconductor) is a gapped system in d dimensions that is characterized by the following two properties:

1. If one brings the TI in contact with a topologically trivial state (e.g. the vacuum), gapless edge states emerge at the boundary (see Fig 2.1).

2. The gapless degrees of freedom are completely robust against disorder. That means we can add any random perturbation or potential of arbitrary strength to the Hamiltonian and the edge states will not develop a gap (as long as the perturbations preserve basic symmetries such as time reversal and do not close the bulk gap).

The most famous example of a TI is the quantum Hall (QH) insulator. In a two-dimensional electron system at low temperatures and high magnetic fields the bulk of the system is insulating while the edges carry quantized current. The gapless fermions at the edge can only move in one direction (since time reversal invariance is broken by the magnetic field) and realize the chiral Luttinger liquid (CLL). Because they can only move in one direction the edge states are "topologically protected" from scattering off disorder and any form of localization is impossible.

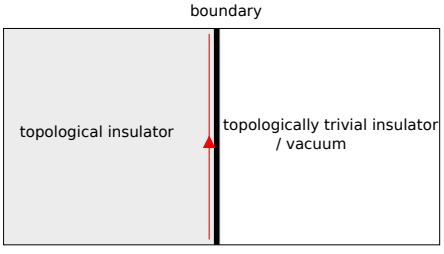

Figure 2.1.: Interface between a TI and a topologically trivial insulator with gapless edge states.

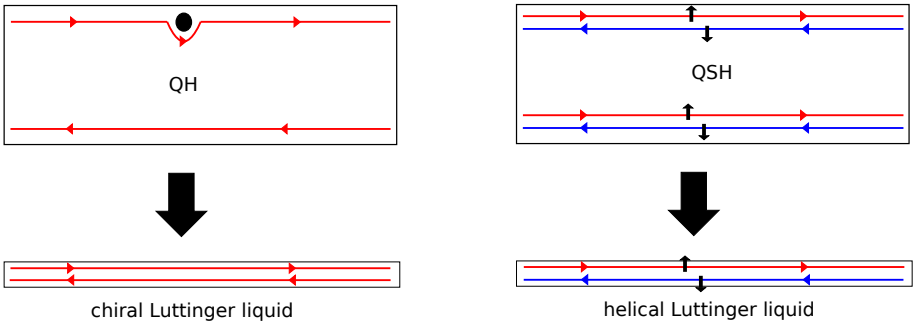

Figure 2.2.: Cartoon of the quantum Hall effect (a) and quantum spin Hall effect (b) as well as the emerging one-dimensional edge states. The dot in figure (a) symbolizes a impurity that does not affect transport due to the topological protection of the edge states.

The system is characterized by a topological invariant, the Chern number ν, that corresponds to the number of edge modes and thus determines the Hall conductivity

$$\sigma_{xy} = \nu \frac{e^2}{h}, \qquad \nu \in \mathbb{Z}. \tag{2.1}$$

The system we are interested in is the quantum spin Hall (QSH) state that exists in band insulators in $d = 2, 3$ dimensions with conserved TRI and strong SOC. These systems can be experimentally realized in HgTe/CdTe quantum wells. Qualitatively one can think of these systems as two copies of the QH state in opposite magnetic field. Therefore, there exist two counterpropagating gapless edge modes that carry opposite spin. Because of the well-defined projection of spin onto the momentum direction the one-dimensional edge state was named helical Luttinger liquid (HLL).

Similar to the QHE the edge states cannot be backscattered by nonmagnetic impurities, because in order to backscatter a right moving particle with spin up the process would have to flip the spin to get a left moving particle with spin down. Therefore, one uses the terminology that edge states are "topologically protected". As in the QHE this protection can be described by a topological invariant. In this case it is a $\mathbb{Z}_2$ number that specifies if there is an even (topologically trivial state) or odd (topologically nontrivial state) number of Kramers partners at the edge. Since each one-dimensional channel carries a conductance quantum e^2/h the edge conductance in the QSH state is:

$$G = 2\frac{e^2}{h}. \tag{2.2}$$

The comparison of QHE and QSHE is depictet in Fig. 2.2.

2.1. Topology in condensed matter systems

In order to get more insight into the role that topology plays in condensed matter systems we start by introducing some concepts of topological materials and their classification, following mostly Ref. [19].

2.1.1. Symmetry classification of Hamiltonians

Consider the gapped Hamiltonian and the corresponding groundstate of a d-dimensional TI. As already discussed, the topological properties we are interested in do not change if we add arbitrary perturbations to the Hamiltonian, as long as we do not close the bulk gap or break any fundamental symmetries. It is clear that these properties are not classified by ordinary symmetries which are represented by unitary operators that commute with

Cartan label	T	C	S	Hamiltonian	G/H (ferm. NLσM)
A(unitary)	0	0	0	U(N)	U(2n)/U(n)× U(n)
AI(orthogonal)	+1	0	0	U(N)/O(N)	Sp(2n)/Sp(n)× Sp(n)
AII(symplectic)	-1	0	0	U(2N)/Sp(2N)	O(2n)/O(n)× O(n)
AIII(chiral unit.)	0	0	1	U(N+M)/U(N)× U(M)	U(n)
BDI (ch. orth.)	+1	+1	1	O(N+M)/O(N)× O(M)	U(2n)/Sp(2n)
CII(ch. sympl.)	-1	-1	1	Sp(N+M)/Sp(N)× Sp(M)	U(2n)/O(2n)
D (BdG)	0	+1	0	SO(2N)	O(2n)/U(n)
C (BdG)	0	-1	0	Sp(2N)	Sp(2n)/U(n)
DIII (BdG)	-1	+1	1	SO(2N)/U(N)	O(2n)
CI (BdG)	+1	-1	1	Sp(2N)/U(N)	Sp(2n)

Table 2.1.: Listed are the ten symmetry classes of single particle Hamiltonians, classified according to the behavior under time reversal ($\mathcal{T}$), charge conjugation ($\mathcal{C}$) and chiral symmetry ($\mathcal{S}$). The column "Hamiltonian" lists the symmetric space of which the quantum mechanical time evolution operator $\exp(itH)$ is an element and the first column is the name of the corresponding space in the mathematical classification by Cartan. The last column lists the (compact sectors of the) target space of the Non linear σ model describing Anderson localization physics in the low energy sector in this symmetry class.

the Hamiltonian. Consider for example spin rotation invariance in the z-direction. If S_z is conserved, the Hamiltonian takes block diagonal form, where the blocks correspond to $S_z = \uparrow, \downarrow$ respectively. Then it is sufficient to study one of the two equivalent copies and S_z symmetry has given us no additional information. This block decomposition can be performed with any operator that commutes with the Hamiltonian. Therefore, one must study the symmetry properties of these blocks of irreducible Hamiltonians in terms of the most basic symmetries. These are time reversal $\mathcal{T}$ and charge conjugation (particle-hole symmetry) $\mathcal{C}$, which are represented by antiunitary operators when acting on the single particle Hilbert space. Every antiunitary operator can be expressed through the combination of a unitary matrix U and complex conjugation $\mathcal{K}$. In our case $\mathcal{T} = U_T\mathcal{K}$ and $\mathcal{C} = U_C\mathcal{K}$. Now consider the Hamiltonian of non-interacting fermions in second quantized language

$$\hat{H} = \sum_{\alpha,\beta} \hat{\psi}^\dagger_\alpha \mathcal{H}_{\alpha,\beta} \hat{\psi}_\beta, \tag{2.3}$$

where α, β are arbitrary quantum numbers and $\mathcal{H}_{\alpha,\beta}$ is the first quantized Hamiltonian. When regularized, i.e. put on a lattice, $\mathcal{H}_{\alpha,\beta}$ is a $N \times N$ matrix. Symmetry under time reversal or charge conjugation can be expressed in terms of $\mathcal{H}$:

$$U_T^\dagger \mathcal{H}^* U_T = +\mathcal{H}, \tag{2.4}$$

$$U_C^\dagger \mathcal{H}^* U_C = -\mathcal{H}. \tag{2.5}$$

It is easy to see that there are 10 ways for a system to respond to the action of time reversal and charge conjugation. Time reversal can either be absent (T=0) or present. If the system is time reversal invariant the antiunitary operator $\mathcal{T}$ can square to ± 1 which one usually denotes by $T = \pm 1$. The same applies for charge conjugation and therefore we are left with $3 \times 3 = 9$ possible classes. To completely specify the behavior under $\mathcal{T}$ and $\mathcal{C}$ one also has to introduce the combined operation $\mathcal{S} = \mathcal{T}\mathcal{C}$ (chiral symmetry), which is unitary. The behavior under $\mathcal{S}$ is always uniquely fixed except for the case $T = C = 0$, where either $S = 0$ or $S = 1$. Here we write $S = 1$ if $\mathcal{S}$ is a symmetry of the Hamiltonian

and $S = 0$ if it is not. The resulting classification of random Hamiltonians is very general and can in particular be used to classify topological states of matter in all dimensions. The 10 symmetry classes are depicted in Tab. 2.1.

2.1.2. Origin of topology in band insulators

For simplicity let us consider a translationally invariant band insulator. Due to the translation invariance Bloch's theorem holds i.e. eigenstates of the Hamiltonian are characterized by the crystal momentum $\mathbf{k} \in$ 1.BZ and the band index n:

$$H(\mathbf{k})|u_n(\mathbf{k})\rangle = \epsilon_n(\mathbf{k})|u_n(\mathbf{k})\rangle. \tag{2.6}$$

We define the projector onto the filled Bloch states for any $\mathbf{k} \in$ 1.BZ as

$$P(\mathbf{k}) = \sum_n^{\text{filled}} |u_n(\mathbf{k})\rangle\langle u_n(\mathbf{k})|. \tag{2.7}$$

It turns out to be more useful to work with the "simplified Hamiltonian" Q($\mathbf{k}$) defined as

$$Q(\mathbf{k}) = 1 - 2P(\mathbf{k}), \tag{2.8}$$

which is obtained from H by assigning to the m filled states the eigenvalue (+1) and to the l empty states the eigenvalue (-1), while leaving the eigenfunctions unchanged. If we are only interested in topological properties, we can always deform $H(\mathbf{k})$ by adding perturbations until it acquires the form of $Q(\mathbf{k})$. Let us see how we can use the matrix Q to identify topological materials by considering the simplest class, class A, where no symmetry conditions are imposed on the Hamiltonian. Then H is an arbitrary Hermitian matrix and Q a unitary matrix $Q \in U(m+l)$. However, because relabelling filled and empty states amongst themselves leaves the physics unchanged, Q is actually a map from the 1.BZ to the so called Grassmannian $U(m+l)/\left[U(m) \times U(l)\right]$:

$$\begin{aligned} Q: \quad \text{1.BZ} &\rightarrow U(m+l)/\left[U(m) \times U(l)\right], \\ \mathbf{k} &\rightarrow Q(\mathbf{k}). \end{aligned} \tag{2.9}$$

Let us summarize. As long as the bulk gap is not closed we can deform the Hamiltonian of the system by adding perturbations until it takes the form of Q. If we want to ask the question how many distinct gapped phases a system posseses this is equivalent to asking how many different maps $Q(\mathbf{k})$ there are that can't be continously deformed into each other. The answer to this question is given by homotopy theory.

Let us quickly review some mathematical basics. In quantum field theory we are mostly interested in mappings from compactified d-dimensional space time[1] into the target manifold M of fields ϕ of the theory, i.e.

$$\begin{aligned} \phi: \quad S^d &\rightarrow M, \\ \mathbf{x} &\rightarrow \phi(\mathbf{x}). \end{aligned} \tag{2.10}$$

We will consider two fields to be equivalent, $\phi_1 \sim \phi_2$, if they can be continously deformed into each other, i.e. if there exists a continous mapping ("homotopy") Φ,

$$\begin{aligned} \Phi: \quad S^d \times [0,1] &\rightarrow M, \\ (\mathbf{x}, a) &\rightarrow \Phi(\mathbf{x}, a), \end{aligned} \tag{2.11}$$

[1] The fields ϕ take values in spacetime $\mathbb{R}^d$. However, to keep the action finite we have to impose the condition $\phi(\mathbf{x}) \rightarrow const.$ for $\|\mathbf{x}\| \rightarrow \infty$. Thus the base manifold of the fields is actually isomorphic to the d-dimensional unit sphere $\mathbb{R}^d \cup \{\infty\} \simeq S^d$.

such that $\Phi(\mathbf{x}, 0) = \phi_1(\mathbf{x})$ and $\Phi(\mathbf{x}, 1) = \phi_2(\mathbf{x})$. The equivalence class containing all fields homotopic to ϕ is denoted by $[\phi]$. The set of all topological equivalence classes $\{[\phi]\}$ of mappings $\phi : S^d \to M$ is called the d-th homotopy group $\Pi_d(M)$. The homotopy groups of the target manifold in the above case are well-known and in d=2,3 dimensions we have

$$\Pi_2(U(m+l)/\left[U(m)\times U(l)\right]) = \mathbb{Z}, \tag{2.12}$$

$$\Pi_3(U(m+l)/\left[U(m)\times U(l)\right]) = \{1\}\,. \tag{2.13}$$

Therefore, for any integer $\nu \in \mathbb{Z}$ there exists a band insulator in d=2 dimensions in symmetry class A. These insulators cannot be continuously deformed into one another without crossing a quantum phase transition. This state of matter is the quantum Hall effect and the topological invariant ν denotes the number of chiral edge modes. When this number changes we indeed have a quantum phase transition - the quantum Hall transition. The fact that the third homotopy group is trivial also tells us that there are no quantum Hall effects in 3 dimensions.

2.2. $\mathbb{Z}_2$ topological insulators

After discussing the general case we will now concentrate on TIs of the class AII with conserved TRI and strong SOI. Because of the great relevance of TRI and SOI we start by reviewing some basic results of quantum mechanics before we delve deeper into physical realisations of $\mathbb{Z}_2$ TI.

2.2.1. Time reversal symmetry

The time reversal operator is an antiunitary operator that squares to (+1) in the case of integer total angular momentum and (-1) in the case of half integer total angular momentum [21]. In our case we are interested in spin 1/2 fermions. Therefore, $\mathcal{T}^2 = -1$ and we choose the representation:

$$\mathcal{T} = i\sigma_y\mathcal{K}. \tag{2.14}$$

Its action can be roughly summarized as $\mathcal{T}(x, k, \sigma) = (x, -k, -\sigma)$. We know from Noether's theorem that any unitary symmetry leads to a conserved quantity (e.g. translation invariance leads to momentum conservation). Although there is no such theorem for antiunitary operators they still have important consequences for the physics of the system. In the case of time reversal, this is Kramer's degeneracy:

"*For systems with half integer total angular momentum each eigenstate of the Hamiltonian is twice degenerate.*"

To show this, consider the eigenvalue problem

$$\hat{H}|E_n\rangle = E_n|E_n\rangle. \tag{2.15}$$

If the Hamiltonian commutes with the time reversal operator $[\hat{\mathcal{T}}, \hat{H}] = 0$, there are two eigenstates $\{|E_n\rangle, \hat{\mathcal{T}}|E_n\rangle\}$ for the energy E_n. We will show that these are distinct states by proof by reductio ad absurdum.
If the two states were the same, they would differ at most by a phase factor

$$\hat{\mathcal{T}}|E_n\rangle = e^{i\delta}|E_n\rangle, \tag{2.16}$$

hence,

$$\hat{\mathcal{T}}^2|E_n\rangle = \hat{\mathcal{T}}e^{i\delta}|E_n\rangle = e^{-i\delta}\hat{\mathcal{T}}|E_n\rangle = |E_n\rangle. \tag{2.17}$$

However, since $\hat{\mathcal{T}}^2 = -1$:

$$\hat{\mathcal{T}}^2|E_n\rangle = -|E_n\rangle. \tag{2.18}$$

This is a contradiction to our assumption that $|E_n\rangle$ is an eigenstate (in particular it is nonzero). Therefore, every energy eigenvalue is at least twice degenerate in the presence of time reversal symmetry and the corresponding eigenkets are orthogonal.

2.2.2. Spin-orbit coupling

Spin-orbit interaction is a contribution to the Hamiltonian of a spinful system that arises naturally in the context of the Dirac equation and describes corrections to the spectrum due to the coupling of the spin of a particle and its motion. Let us consider spin-orbit coupling as a weak perturbation. In perturbation theory around $\mathbf{k} = 0$ one expects the lowest terms coupling to the spin to be linear in $\mathbf{k}$,

$$\hat{H}_{SO} = -\mathbf{B}(\mathbf{k})\boldsymbol{\sigma}. \tag{2.19}$$

Time reversal symmetry requires $\mathbf{B}(-\mathbf{k}) = -\mathbf{B}(\mathbf{k})$. If the system is in addition to that also inversion symmetric, $\mathbf{B}(\mathbf{k}) = \mathbf{B}(\text{-}\mathbf{k})$, the only possible solution is $\mathbf{B}(\mathbf{k}) = 0$. Therefore, in order for the spin-orbit term to be nonzero, inversion symmetry has to be broken.[2]

There are two possibilities to break inversion symmetry in a system:

- Structural inversion asymmetry (SIA) and
- Bulk inversion asymmetry (BIA).

In heterostructures the confinement potential and the band edge variations (different materials have different band gaps) break the inversion symmetry. This is called SIA. The corresponding perturbation to the Hamiltonian is the Rashba SOI. This term can be derived from the microscopic 8×8 Kane model [22, 23] but here we will only motivate it heuristically. The Rashba effect is a consequence of inversion symmetry breaking perpendicular to the 2d plane, e.g. by an electric field $\mathbf{E} = E_0\mathbf{e}_z$. Due to relativistic corrections an electron moving with velocity $\mathbf{v}$ in the electric field will experience an effective magnetic field in its rest frame,

$$\mathbf{B} = -\frac{1}{c^2}(\mathbf{v} \times \mathbf{E}) = -\frac{E_0}{mc^2}(\mathbf{k} \times \mathbf{e}_z). \tag{2.20}$$

The magnetic field couples to the electron's spin as

$$\hat{H}_R = \mu_B \boldsymbol{\sigma}\mathbf{B} = -\frac{E_0\mu_B}{mc^2}\boldsymbol{\sigma}(\mathbf{k} \times \mathbf{e}_z). \tag{2.21}$$

We also have to include a factor $1/2$ due to the Thomas precession. Therefore, we arrive at the Rashba term

$$\hat{H}_R = \alpha(\boldsymbol{\sigma} \times \mathbf{k})\,\mathbf{e}_z, \tag{2.22}$$

where $\alpha = \mu_B E_0/(2mc^2)$. A neat way of picturing this is to write the Rashba term as Zeeman splitting with a momentum dependend magnetic field as in Eq. (2.19):

$$\hat{H}_R = -\mathbf{B}(\mathbf{k})\boldsymbol{\sigma}, \qquad \mathbf{B}(\mathbf{k}) = \alpha(-k_y, k_x, 0). \tag{2.23}$$

[2] Of course we could also break TRI by applying a magnetic field. However, here we are exclusively interested in systems with conserved TRI and therefore we will not consider this case.

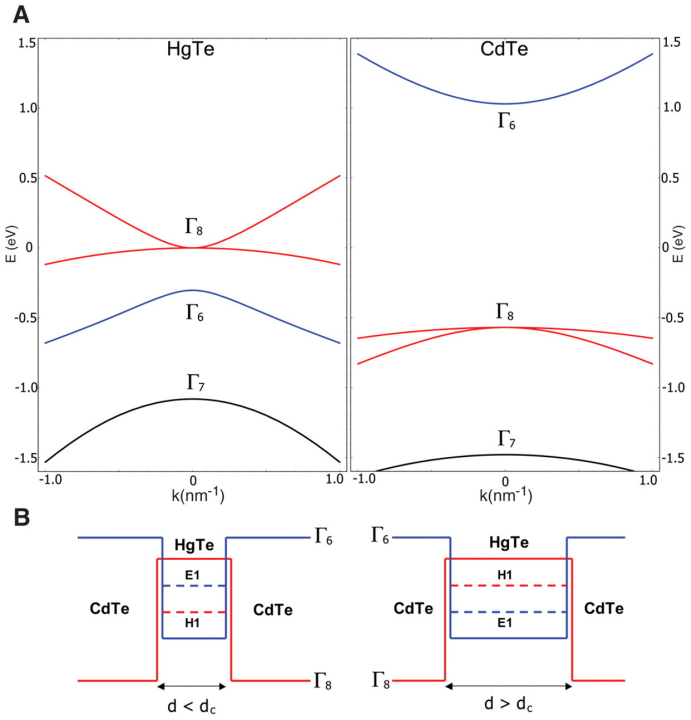

Figure 2.3.: (A) bulk bandstructure of HgTe and CdTe and (B) schematic picture of quantum well energy levels for different thicknesses (from [4]). For quantum well thickness $d < d_c$ the system is in the topologically trivial phase, while gapless edge modes emerge at the boundary between HgTe and CdTe for $d > d_c$.

Note that while this simple argument can construct the correct form of the Hamiltonian, the magnitude of the prefactor α is actually two orders smaller than the correct microscopic result.

It is also possible that the intrinsic crystal structure of the bulk breaks inversions symmetry (BIA). For example this is the case in the zincblende structure. The corresponding SOI term is called Dresselhaus term and is given by

$$H_D = \beta \left(k_x \sigma_x - k_y \sigma_y \right). \tag{2.24}$$

2.2.3. Physical realisations of $\mathbb{Z}_2$ topological insulators

The QSH state was first discovered in HgTe/CdTe quantum wells. In this section we review the basic electronic structure of the semiconductors HgTe and CdTe and derive a simple model, following Ref. [4], to describe the physics of the subbands of HgTe/CdTe quantum wells that are important for the QSH effect.

For both HgTe and CdTe the important bands near the Fermi level are close to the center of the Brillouin zone (Γ point). Quite generally one can picture the bands in solids as combinations of atomic orbitals of the constituent atoms. If two atoms are brought close together, the orbitals can overlap in two ways to form bonding or antibonding molecular orbitals. Normally, one concentrates on the antibonding s-type orbitals that make up the valence band and the bonding p-type orbitals that build the conduction band. Near the Γ point these are twice (s-type) or sixfold (p-type) degenerate due to the spin degree of freedom. If we take into account spin-orbit coupling, the bands become mixed and are characterized by their total angular momentum j i.e. we have quantum numbers (j, m_j, l, s).

The conduction band consists of the doublet $|\Gamma_6, j = 1/2, m_j = \pm 1/2\rangle$[3] while the valence band is made of the quadruplet $|\Gamma_8, j = 3/2, m_j = \pm 1/2\rangle$, $|\Gamma_8, j = 3/2, m_j = \pm 3/2\rangle$ and the doublet $|\Gamma_7, j = 1/2, m_j = \pm 1/2\rangle$. The $j = 3/2$ bands are degenerate at the Γ point and have different curvatures (effective masses) away from it. Therefore, they are referred to as heavy and light hole bands. The Γ_7 band is split off by SOI and will be disregarded in the following.

In quantum wells grown in [001] direction the spherical symmetry is broken down to a axial rotation symmetry in the plane. The six bands therefore combine to form the spin up and down states of the quantum well subbands E1, H1 and L1. The L1 subband is separated from the other two and we will neglect it, leaving an effective four band model. At the Γ point with in-plane momentum $k_\parallel = 0$, m_j is still a good quantum number. The quantum well states $|E1, m_j = \pm 1/2\rangle$ are then linear combinations of states $|\Gamma_6, j = 1/2, m_j = \pm 1/2\rangle$ and $|\Gamma_8, j = 1/2, m_j = \pm 1/2\rangle$ while $|H1, m_j = \pm 3/2\rangle$ are made of $|\Gamma_8, j = 3/2, m_j = \pm 3/2\rangle$, respectively. Therefore, the quantum well states are given by the ordered basis $\{|E1, \uparrow\rangle, |H1, \uparrow\rangle, |E1, \downarrow\rangle, |H1, \downarrow\rangle\}$.[4]

Note that $|\Gamma_6, m_j = \pm 1/2\rangle$ has even parity and $|\Gamma_8, m_j = \pm 3/2\rangle$ has odd parity under two-dimensional spatial inversion. Therefore, a matrix element coupling the two states has to have an odd power of momentum $\mathbf{k}$.

These symmetry arguments lead to the following model:

$$\begin{aligned} H(k_x, k_y) = &\begin{pmatrix} h(\mathbf{k}) & 0 \\ 0 & h^*(-\mathbf{k}) \end{pmatrix}, \\ h(\mathbf{k}) = &\epsilon(\mathbf{k})\mathbb{1}_2 + d_i(\mathbf{k})\sigma_i, \end{aligned} \tag{2.25}$$

where

$$\epsilon(\mathbf{k}) = C - D(k_x^2 + k_y^2), \qquad d_i(\mathbf{k}) = (Ak_x, -Ak_y, M(\mathbf{k})), \tag{2.26}$$

$$M(\mathbf{k}) = M - B(k_x^2 + k_y^2). \tag{2.27}$$

The important point is that the band structure in HgTe is inverted because of the large SOC in heavy materials like Hg. That means that the Γ_8 lies above the Γ_6 band in this material. For small quantum well thicknesses the heterostructure should behave as CdTe with normal band ordering. However, as we increase the thickness above some critical value d_c, the bandstructure will be determined by the properties of HgTe and become inverted (see Fig. 2.3). Therefore, we have a quantum phase transition as a function of quantum well thickness. In the inverted regime the bands naturally have to cross at the boundaries which leads to the emergence of massless edge states with linear dispersion described by the helical LL. This can be shown explicitely by solving the BHZ model with appropriate boundary conditions [24]. In particular the edge states emerge irrespective of the boundary conditions imposed, which underlines their topological nature.

2.2.4. Effective model for the HLL

The 1D helical system is translation invariant and momenta k are thus good quantum numbers for the eigenstates. Furthermore, the spin degree of freedom of the excitations

[3] The notation Γ_6, Γ_8 etc. originates from the group theoretical classification of irreducible representations of the crystal symmetries which determine the way wavefunctions transform at the Γ point.

[4] For Γ_6 states we have l=0 which implies that $m_j = \pm 1/2$ is the same as $s = \pm 1/2$. On the other hand, for Γ_8 l=1 and $m_j = \pm 3/2$ is also the same as $s = \pm 1/2$ therefore we introduced the notation $s = 1/2 = \uparrow$ and $s = -1/2 = \downarrow$.

is frozen out because each chirality has a well-defined spin direction. The effective low energy theory for the edge excitations is thus that of free spinless fermions,

$$\hat{H}_0 = \frac{1}{L} \sum_{k,\eta=R,L} \eta\, k\, \hat{\psi}^\dagger_{k,\eta} \hat{\psi}_{k,\eta}, \tag{2.28}$$

where $\hat{\psi}_{k,\eta}$ are fermionic operators that annihilate right or left moving excitations respectively and $\eta = +1(R), -1(L)$ denotes chirality. If we assume that time reversal symmetry holds, Kramer's theorem ensures that for any k there exist two orthogonal eigenstates, created by fermionic operators $\hat{\psi}^\dagger_{R,k}$ and $\hat{\psi}^\dagger_{L,-k}$ which are related by time reversal $\mathcal{T}$, e.g.

$$\mathcal{T} \begin{pmatrix} \hat{\psi}_{k,R} \\ \hat{\psi}_{k,L} \end{pmatrix} = i\sigma_y \mathcal{K} \begin{pmatrix} \hat{\psi}_{k,R} \\ \hat{\psi}_{k,L} \end{pmatrix} = \begin{pmatrix} \hat{\psi}_{-k,L} \\ -\hat{\psi}_{-k,R} \end{pmatrix}. \tag{2.29}$$

We also take into account interaction in the density-density channel and we model impurities by a local potential that couples to the electronic density:

$$\hat{H}_{\text{int}} = \frac{1}{L} \sum_{kqp} \sum_{\sigma\sigma'} V(q) \hat{\psi}^\dagger_{k,\sigma} \hat{\psi}_{k-q,\sigma} \hat{\psi}^\dagger_{p,\sigma'} \hat{\psi}_{p+q,\sigma'}, \tag{2.30}$$

$$\hat{H}_{\text{imp}} = \frac{1}{L} \sum_{k,p} \sum_{\sigma} U(k-p) \hat{\psi}^\dagger_{\sigma,k} \hat{\psi}_{\sigma,p}. \tag{2.31}$$

For a generic helical liquid spin rotation invariance around the z-direction will be broken either by SIA or BIA effects. We therefore formulate the problem in the chiral basis (R, L) in which the free Hamiltonian is diagonal. In order to perform this rotation we follow Ref. [11] and derive the rotation matrix from symmetry arguments. The field operators $\hat{\psi}_{\sigma,k}$ of an electron with momentum k and spin projection σ along the z-axis are related to the chiral operators $\hat{\psi}_{\eta,k}$ by a momentum dependant SU(2) matrix B_k,

$$\begin{pmatrix} \hat{\psi}_{\uparrow,k} \\ \hat{\psi}_{\downarrow,k} \end{pmatrix} = B_k \begin{pmatrix} \hat{\psi}_{R,k} \\ \hat{\psi}_{L,k} \end{pmatrix}. \tag{2.32}$$

To preserve fermionic commutation relations the matrix has to be unitary $B^\dagger_k B_k = \mathbb{1}$. Moreover, time reversal invariance entails the symmetry $B_k = B_{-k}$. Because of these constraints the leading terms in B_k for small $k \ll k_0$ can be written as:

$$B_k = \begin{pmatrix} 1 - \frac{k^4}{2k_0^4} & -\frac{k^2}{k_0^2} \\ \frac{k^2}{k_0^2} & 1 - \frac{k^4}{2k_0^4} \end{pmatrix}, \qquad B_k^{-1} = \begin{pmatrix} 1 - \frac{k^4}{2k_0^4} & \frac{k^2}{k_0^2} \\ -\frac{k^2}{k_0^2} & 1 - \frac{k^4}{2k_0^4} \end{pmatrix}. \tag{2.33}$$

Here, k_0^{-1} is the typical distance an electron can travel while retaining its spin-direction and higher order terms in k/k_0 are supposed to be small. Finally, we neglect the momentum dependance of the interaction and impurity potentials and assume $U(q) = U_0$ and $V(q) = V_0$. In the case of interactions this is justified if the potential is well screened by external media (e.g. external gates) and for impurities we model the potential to be short-range in real space.

If we substitute the relation (2.32): $\hat{\psi}_{\sigma,k} = \sum_{\eta=R,L} B_{k,\sigma\eta} \hat{\psi}_{\eta,k}$ into the interaction Hamiltonian we get:

$$\hat{H}_{\text{int}} = \frac{1}{L} \sum_{kqp} \sum_{\eta_1\eta_2\eta_3\eta_4} V(q) \left[B^\dagger_k B_{k-q}\right]_{\eta_1,\eta_2} \left[B^\dagger_p B_{p+q}\right]_{\eta_3,\eta_4} \hat{\psi}^\dagger_{k,\eta_1} \hat{\psi}_{k-q,\eta_2} \hat{\psi}^\dagger_{p,\eta_3} \hat{\psi}_{p+q,\eta_4}. \tag{2.34}$$

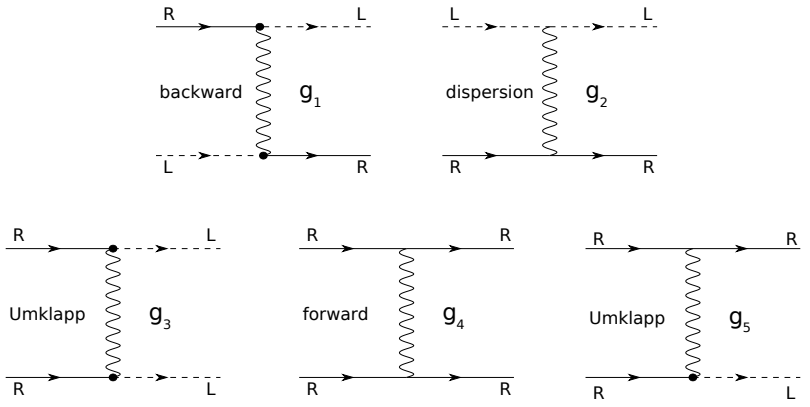

Figure 2.4.: G-ology of interaction terms in the HLL. Fat vertices denote chirality changes that have an additional prefactor $\eta_{\text{in}}(k_{\text{in}}^2 - k_{\text{out}}^2)$.

Therefore, we have to calculate the product

$$B_k^\dagger B_p \approx \begin{pmatrix} 1 & \frac{k^2-p^2}{k_0^2} \\ -\frac{k^2-p^2}{k_0^2} & 1 \end{pmatrix}. \tag{2.35}$$

Here, we neglected higher order terms in k/k_0. This product can be written in the form

$$\left[B_k^\dagger B_p\right]_{\eta,\eta'} = \delta_{\eta,\eta'} + \eta\, \delta_{\bar{\eta},\eta'}(k^2 - p^2), \tag{2.36}$$

where we use the notation $\bar{R} = L$ and vice versa. Inserting (2.36) into (2.34) we get after some simple algebra:

$$\begin{aligned} \hat{H}_{\text{int}} =& \frac{V_0}{L} \sum_{kqp} \sum_{\eta\eta'} \Big[\hat{\psi}_{k,\eta}^\dagger \hat{\psi}_{p,\eta'}^\dagger \hat{\psi}_{p+q,\eta'} \hat{\psi}_{k-q,\eta} \\ &+ 2\eta \frac{\left(k^2 - (k-q)^2\right)}{k_0^2} \hat{\psi}_{k,\eta}^\dagger \hat{\psi}_{p,\eta'}^\dagger \hat{\psi}_{p+q,\eta'} \hat{\psi}_{k-q,\bar{\eta}} \\ &+ \eta\eta' \frac{\left(k^2 - (k-q)^2\right)}{k_0^2} \frac{\left(p^2 - (p+q)^2\right)}{k_0^2} \hat{\psi}_{k,\eta}^\dagger \hat{\psi}_{p,\eta'}^\dagger \hat{\psi}_{p+q,\bar{\eta}'} \hat{\psi}_{k-q,\bar{\eta}} \Big]. \end{aligned} \tag{2.37}$$

Note that we already normal ordered the Hamiltonian (any two particle processes that are created due to the anticommutation relations can be absorbed in the chemical potential). The different terms of the interaction Hamiltonian can be grouped analogously to the g-ology of a conventional LL, which is shown explicitely in Appendix A. However, now we have an additional Umklapp term that backscatters only one particle. For the purpose of this thesis it will be called g_5 term. All terms are depicted in Fig: 2.4.

Lastly, the impurity part in the chiral basis reads

$$\hat{H}_{\text{imp}} = \frac{U_0}{L} \sum_{k,p} \sum_{\eta} \left(\hat{\psi}_{k,\eta}^\dagger \hat{\psi}_{p,\eta} + \eta\, \frac{k^2 - p^2}{k_0^2} \hat{\psi}_{k,\eta}^\dagger \hat{\psi}_{p,\bar{\eta}} \right). \tag{2.38}$$

One thing should be mentioned at this point. The parameters V_0 and U_0 should be considered as effective couplings of the low energy theory after integrating out all degrees of freedom above the UV cutoff, which is given by the bulk gap. Therefore, renormalization effects due to high lying states are already incorporated into the coupling constants and do not affect the physics apart from that.

Let us summarize:

1. The effective low energy model for the gapless edge modes of the QSH insulator is that of spinless one-dimensional fermions in the chiral basis.
2. Each chirality changing process comes with a factor $\eta_{\text{in}}(k_{\text{in}}^2 - k_{\text{out}}^2)$. These factors embody the topological protection the edge modes enjoy. First, the momentum factors change the scaling dimensions of the operators making them irrelevant in the RG sense. Therefore, chirality changing processes cannot open a gap. Second, states with identical momentum but opposite spin are orthogonal due to Kramer's theorem. The factor $(k_{\text{in}}^2 - k_{\text{out}}^2)$ thus ensures that there is no scattering between Kramers partners in the helical basis either.

Before analysing the model further, we have to gain a better understanding of one-dimensional physics in general. Therefore, the next section will be devoted to a survey of quantum physics in one dimension.

3. One-dimensional systems

One-dimensional systems of interacting particles have fascinated theorists for more than 50 years now. The profound effect of dimensionality drastically changes the behavior of particles compared to higher dimensional counterparts. On the one hand, the one-dimensional character leads to the possibility to solve some systems exactly while on the other hand the interplay of interactions, disorder and the low dimensionality leads to highly nontrivial correlation effects. In the following we will review, mostly following [16], the powerful techniques that are at our disposal to solve problems in one dimension. For a more detailed discussion see Refs. [16, 25, 26].

3.1. A word on conformal symmetry and the renormalization group

3.1.1. The renormalization group

Here we will present only a very basic introduction to the renormalization group. For further studies we refer to the literature [27, 28].

In condensed matter physics the *renormalization group* (RG) is a general term for methods that allow systematic investigations of a system's behavior under scale changes. In particular one is often interested in obtaining an effective low energy theory by integrating out high energy degrees of freedom.
A field theory is by definition a physical model defined on the continuum instead of a lattice. However, quantum fluctuations being at work at all scales [1], the theory is prone to all sorts of UV or IR divergences. Therefore, a purely continous theory makes no sense and we have to introduce an energy cutoff Λ that marks the range of applicability of the model. Such a *regularization* invariably introduces a length scale Λ^{-1}, which is part of the theory as one of the parameters along with other coupling constants, masses, velocities and so on. One must keep in mind though that a change in the cutoff Λ (through tracing out the high energy degrees of freedom) is accompanied by a change of all other parameters. Thus a field theory is not characterized by a fixed set of coupling constants but by a renormalization group trajectory in parameter space, which traces the change of parameters as the cutoff is lowered.

[1] that means that the fields $\phi(\mathrm{k})$ take values at all momenta $k \in \{-\infty, \infty\}$.

Let us outline how a generic (momentum) RG step is done. The problem we face, is that the theory is in general unaware of any clear cut separation of "fast" and "slow" [2] degrees of freedom. To nonetheless implement our scheme of integrating out "fast" modes we have to artificially define an energy $\Lambda^{(1)} = b^{-1}\Lambda$, where b>1, that separates slow ($\omega < \Lambda/b$) and fast ($\Lambda/b < \omega < \Lambda$) variables. Now we integrate out the fast fields and arrive at an effective action for the slow fields. The result may be twofold, either the algebraic structure of the action is changed completely or we get a new action that is identical to previous one except for (i) a different set of coupling constants and (ii) a decreased energy cutoff $\Lambda \to \Lambda^{(1)} = b^{-1}\Lambda$. In the later case the theory is called *renormalizable*. To compare the set of coupling constants with the previous ones we have to rescale momenta and energies, such that they have the same cutoff as before $(k, \omega) \to (bk, b\omega)$[3]. Now one can iterate the procedure until the natural cutoff of the theory is reached (e.g. if temperature is the highest energy scale of the problem, the cutoff will be T). However, the utility of the RG relies on the recursive reproduction of the model at each step: one step alone already encodes all information about the renormalization properties of the model. These properties are condensed in the *generalized β function* which describes the flow of the set of parameters $\mathbf{g}$ under the change of control parameter

$$\beta(\mathbf{g}) = \frac{d\mathbf{g}}{d \ln b}. \tag{3.1}$$

Most important is the notion of *fixed point*, i.e. a point in parameter space $\mathbf{g}^*$ that is unaffected by the renormalization step ($\beta(\mathbf{g}^*) = 0$). Exactly at a fixed point the system is scale invariant. However, every system has at least one intrinsic length scale, the correlation length ξ. Consequently, at a fixed point we must have either $\xi = 0$ or $\xi = \infty$. On the other hand, a diverging correlation length is the hall mark of a second order phase transition. Therefore, we can indentify fixed points as critical points of the underlying physical model.

In the vicinity (in a pertubative sense) of a fixed point the terms in the action, characterized by their coupling constants, fall into three categories: relevant, irrelevant and marginal. Relevant parameters grow algebraically under renormalization, irrelevant parameters decrease algebraically and marginal parameters undergo logarithmic variations.

3.1.2. Conformal field theory

Conformal field theory (CFT) provides some very powerful methods for solving two-dimensional statistical mechanics problems. We therefore feel the need to introduce some of the basic concepts. For an exhaustive overview of the field we refer to the literature [29, 30].

By definition the conformal group is a subgroup of coordinate transformations that leaves the metric invariant up to a scale change, i.e.

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{3.2}$$

That means the transformation conserves angles, but not lengths. In d dimensions these transformations form a finite group with (d+1)(d+2)/2 independant parameters. A theory is called conformally invariant if its action is invariant under the conformal group.

In 2D, conformal transformations coincide with analytic coordinate transformations. It is thus natural to define complex coordinates in terms of real space coordinate x and

[2] with $\hbar \equiv 1$ energy and frequency are measured in the same units. Thus "fast" is equivalent to high energies and "slow" to low energies.

[3] In a Lorentz invariant theory it makes sense to scale energy and momentum with the same factor.

imaginary time $\tau = it$,

$$z = v\tau - ix, \qquad \bar{z} = v\tau + ix. \tag{3.3}$$

Here v is velocity and we have the following correspondence for the derivatives:

$$\partial_z = -\frac{i}{2}\left(\frac{1}{v}\partial_t - \partial_x\right), \qquad \partial_x = -i\left(\partial_z - \partial_{\bar{z}}\right), \tag{3.4}$$

$$\partial_{\bar{z}} = -\frac{i}{2}\left(\frac{1}{v}\partial_t + \partial_x\right), \qquad \partial_t = iv\left(\partial_z + \partial_{\bar{z}}\right). \tag{3.5}$$

Indeed under the transformation

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}) \tag{3.6}$$

the line element changes as

$$ds^2 = dzd\bar{z} \to \left|\frac{df}{dz}\right|^2 dzd\bar{z}. \tag{3.7}$$

Therefore, comparing this expression with Eq. (3.2), we have $\Omega = |df/dz|^2$.

The peculiarity of 2D is now that one cannot only define a global conformal group as in other dimensions but also a local conformal algebra.[4] The class of analytic functions that are invertible and defined on the whole Riemann sphere (i.e. have no branch cuts) make up the global conformal group. In terms of complex coordinate z, global conformal transformations have the form

$$z \to f(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1, \tag{3.8}$$

where a,b,c,d are complex numbers. However, since any analytic function is a conformal mapping one has an infinite algebra of local conformal transformations, that have all the properties of conformal maps except that they are not one to one. This feature of 2D CFT is what allows a complete solution of conformal field theories, even in certain circumstances that apparently break scale invariance. For instance, the complete complex plane (the space time used with imaginary time at zero temperature) may be mapped onto a cylinder of circumference L via the mapping $f(z) = e^{2\pi z/L}$. This allows for the calculation of correlation functions in a system with a macroscopic length scale (a finite size L at zero temperature, or a finite temperature $\beta = L/v$ in an infinite system) from the known solution in a scale-invariant situation.

Furthermore, the behavior of operators under the conformal group gives us information about their transformation properties under scale changes, which is useful in combination with an RG treatment. A local operator belonging to a CFT is called quasi-primary if it scales under a conformal transformation like

$$O'(\alpha z, \bar{\alpha}\bar{z}) = \alpha^{-h}\bar{\alpha}^{-\bar{h}}O(z, \bar{z}). \tag{3.9}$$

The constants $h, \bar{h}$ are called the right-and left-conformal dimensions, respectively and are a fundamental property of the operator O.
Under plain dilation ($\alpha = \bar{\alpha}$) the operator behaves as $O'(\alpha z, \alpha\bar{z}) = \alpha^{-\Delta}O(z, \bar{z})$, where $\Delta \equiv h + \bar{h}$ is the ordinary scaling dimension of the operator. Under a rotation ($\alpha = e^{i\theta}$, $\bar{\alpha} = e^{-i\theta}$) the operator transforms as $O'(e^{i\theta}z, e^{-i\theta}\bar{z}) = e^{is\theta}O(z, \bar{z})$, where $s \equiv h - \bar{h}$ is the conformal spin of the operator. In general the scaling dimension appears directly in the two point correlation function of the operator which is fixed by scaling arguments up to a multiplicative constant:

$$\langle O(z, \bar{z})O(0, 0)\rangle = \frac{1}{z^{2h}\bar{z}^{2\bar{h}}}. \tag{3.10}$$

[4] Strictly speaking the local conformal transformations don't constitute a group since they are not invertible on the whole Riemann sphere $\mathbb{C} \cup \infty$. Thus we use the word algebra here.

3.1.3. Effect of pertubations

Consider the perturbed action of a one-dimensional system

$$S = S_0 + g \int \mathrm{d}x\mathrm{d}t\, O(x,t) \tag{3.11}$$

where S_0 is the fixed-point action and O is an operator of scaling dimension Δ. Let us now perform a single RG step, that is we change the real space cutoff Λ^{-1} to a higher value $b\Lambda^{-1}$, where $b > 1$. In order to compare the new action with the previous one, we have to rescale spacetime as $(x,t) = (b^{-1}x', b^{-1}t')$, where (x,t) now have the same cutoff as before. Thus

$$\begin{aligned} S' =& S_0 + g \int \mathrm{d}x'\mathrm{d}t'\, O(x',t') && (3.12)\\ =& S_0 + g \int \mathrm{d}(bx)\mathrm{d}(bt)\, O(bx,bt) && (3.13)\\ =& S_0 + g\, b^{2-\Delta} \int \mathrm{d}x\mathrm{d}t\, O(x,t). && (3.14) \end{aligned}$$

Therefore, we can read off the change of the coupling constant,

$$g'(b) = g\, b^{2-\Delta} = g(0)\, e^{(2-\Delta)\ln(b)}. \tag{3.15}$$

Let us assume that $b = 1 + \epsilon$ ($\epsilon \ll 1$) is close to unity (which means we "shave off" an infinitesimal layer in momentum space in each RG step) , i.e. $\ln(b) \approx \epsilon$. Then we can encapsulate the information from the RG step in the differential equation $\frac{g'-g}{\epsilon} \approx \frac{dg}{d\epsilon} = g(2-\Delta) \approx \frac{dg}{d\ln(b)}$. In other words the β function is given by

$$\beta(g) = \frac{dg}{d\ln(b)} = g\,(2-\Delta). \tag{3.16}$$

The sign of the beta function tells us if a coupling constant grows ($\beta > 0$) or decreases ($\beta < 0$) under the RG. Thus, the pertubation is relevant if $\Delta < 2$, irrelevant if $\Delta > 2$ and marginal if $\Delta = 2$. A relevant operator is typically the source of a gap in the low-energy spectrum while irrelevant operators only change the properties of the theory in a perturbative sense.

Let us end the section with some remarks:

- By adding a pertubation that is not invariant under scale changes we break conformal symmetry. However, we may still study the effect of weak coupling in the CFT framework if we don't depart too far from the fixed point. Especially, if the operator is irrelevant.

- The above analysis does not work for operators with conformal spin. If one fuses two operators with conformal spin together, an operator with lower scaling dimension can be generated. These operators would appear in the action under a RG step and have to be included by hand in order to describe the full theory.

3.2. Models for one-dimensional fermions and bosons

3.2.1. The Gaussian model

The model describing free massless bosons in one dimension is called the Gaussian model

$$\begin{aligned} S_0 =& \frac{1}{2} \int \mathrm{d}x\mathrm{d}\tau\, \left[v^{-1} \left(\partial_\tau \varphi\right)^2 + v \left(\partial_x \varphi\right)^2 \right] && (3.17)\\ \overset{\text{IP}}{=}& -\frac{1}{2} \int \mathrm{d}x\mathrm{d}\tau\, \varphi \left(v^{-1}\partial_\tau^2 + v\partial_x^2 \right) \varphi, && (3.18) \end{aligned}$$

where $\varphi(x,\tau)$ is a real scalar field and v describes the velocity. Since the action is quadratic we can immediately read off the equation governing the propagator:

$$-\left(v^{-1}\partial_\tau^2 + v\partial_x^2\right) G(x,\tau,x',\tau') = \delta(x-x')\delta(\tau-\tau'). \tag{3.19}$$

In complex coordinates this is the two-dimensional poisson equation

$$-4v\partial_z\partial_{\bar{z}}G(z,\bar{z}) = \delta(z)\delta(\bar{z}). \tag{3.20}$$

Its solution is well-known and for a disk of radius R in (x,τ) space and a short distance cutoff a_0 we get

$$G(z,\bar{z}) = \frac{1}{4\pi}\ln\left(\frac{R^2}{z\bar{z}+a_0^2}\right) \overset{a_0\to 0}{=} -\frac{1}{4\pi}\ln(z\bar{z}) + \frac{1}{2\pi}\ln(R). \tag{3.21}$$

In the following we drop the second term, which basically defines an overall length scale. Thus we see that at the level of correlation functions the boson φ decomposes into a holomorphic and an antiholomorphic part,

$$\langle\varphi(z,\bar{z})\varphi(0,0)\rangle = -\frac{1}{4\pi}\ln(z\bar{z}) = -\frac{1}{4\pi}\ln(z) - \frac{1}{4\pi}\ln(\bar{z}). \tag{3.22}$$

This corresponds to the decomposition of the field φ into two chiral boson fields $\varphi(z,\bar{z}) = \phi_R(z) + \phi_L(\bar{z})$ with propagators

$$\langle\phi_R(z)\phi_R(0)\rangle = -\frac{1}{4\pi}\ln(z), \tag{3.23}$$

$$\langle\phi_L(\bar{z})\phi_L(0)\rangle = -\frac{1}{4\pi}\ln(\bar{z}). \tag{3.24}$$

It is customary to further define the field $\Pi(x,\tau) = \frac{\delta\mathcal{L}}{\delta\partial_\tau\varphi} = v^{-1}\partial_\tau\varphi$ conjugate to φ. The corresponding Hermitian operators in the Hamiltonian formalism obey the canonical equal time commutation rules

$$\left[\hat{\varphi}(x),\hat{\Pi}(x')\right] = i\delta(x-x'), \tag{3.25}$$

$$\left[\hat{\varphi}(x),\hat{\varphi}(x')\right] = \left[\hat{\Pi}(x),\hat{\Pi}(x')\right] = 0. \tag{3.26}$$

Finally, we define the dual boson[5] $\theta(x,\tau)$ by the relation $\partial_x\theta = -\Pi$. This basic definition implies the nonlocal commutation relation

$$\left[\hat{\varphi}(x),\hat{\theta}(x')\right] = -i\Theta(x-x'), \tag{3.27}$$

where Θ is the Heaviside step function. The representation of θ in terms of chiral bosons is $\theta = \phi_R - \phi_L$.

For future referrence let us define the exponentials of boson fields $e^{i\alpha\phi}$.[6] To avoid divergencies in the theory exponential operators of this form (called vertex operators) have to be normal ordered. Thus, vertex operators don't multiply like ordinary exponentials but according to

$$e^{i\alpha\phi(z)}e^{i\beta\phi(z')} = e^{i\alpha\phi(z)+i\beta\phi(z')}e^{-\alpha\beta\langle\phi(z)\phi(z')\rangle} \tag{3.28}$$

which can be shown using the Campbell-Baker-Hausdorff-Formula [16].

[5] The dual field θ has the same action as φ and consequently the same correlation functions and commutation relations.

[6] $\phi(z)$ is an arbitrary bosonic field.

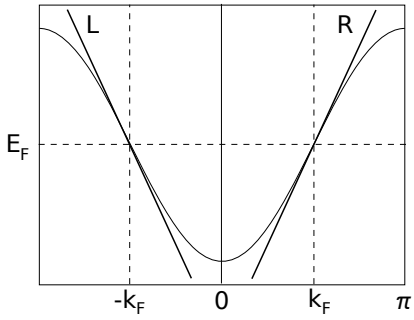

Figure 3.1.: Linearisation of the spectrum around the Fermi points.

The correlation function of vertex operators in an infinite system,

$$\langle e^{i\alpha_1\phi(z_1)}e^{i\alpha_2\phi(z_2)}\dots e^{i\alpha_N\phi(z_N)}\rangle\,, \tag{3.29}$$

is nonzero only if $\sum_{i=1}^{n}\alpha_N = 0$. This is known in the literature as the neutrality condition.

Vertex operators are primary fields. Their conformal dimensions can be obtained from their correlator

$$\langle e^{i\alpha\varphi(z,\bar{z})}e^{-i\alpha\varphi(0,0)}\rangle = \langle e^{i\alpha(\varphi(z,\bar{z})-\varphi(0,0))}\rangle\, e^{\alpha^2\langle\varphi(z,\bar{z})\varphi(0,0)\rangle} = (z\bar{z})^{-\frac{\alpha^2}{4\pi}}\,. \tag{3.30}$$

Therefore, the conformal dimensions of the operator are

$$e^{i\alpha\varphi(z,\bar{z})} : \big(h(\alpha),\bar{h}(\alpha)\big) = \left(\frac{\alpha^2}{8\pi},\frac{\alpha^2}{8\pi}\right). \tag{3.31}$$

3.2.2. The Dirac equation

Free one-dimensional fermions are described by the massless (1+1) dimensional Dirac equation. To show this, we consider the action of free spinless electrons with quadratic spectrum in one dimension and derive their low energy effective theory. The action reads

$$S_0 = \int \mathrm{d}x\mathrm{d}\tau\,\left[\bar{\psi}(x,\tau)\left(\partial_\tau - \frac{\partial_x^2}{2m}\right)\psi(x,\tau)\right], \tag{3.32}$$

where $\bar{\psi},\psi$ are Grassmann variables. For small energies the relevant electrons lie near the Fermi points and we expand our fields in their vicinity, $\psi_{k_F+q}\equiv\psi_R(q)$ and $\psi_{-k_F+q}\equiv\psi_L(q)$ (see Fig. 3.1). After Fourier transforming we get the decomposition

$$\psi(x,\tau) = \psi_R(x,\tau)e^{ik_Fx} + \psi_L(x,\tau)e^{-ik_Fx}, \tag{3.33}$$

where $\psi_{R(L)}$ are smooth fields on the length scale k_F^{-1}. In the Hamiltonian formalism these correspond to operators with commutation rules

$$\{\hat{\psi}_\eta(x),\hat{\psi}^\dagger_{\eta'}(x')\} = \delta(x-x')\delta_{\eta,\eta'}. \tag{3.34}$$

Using (3.33) in (3.32) and neglecting oscillatory terms and higher order derivatives in ψ we arrive at

$$S_0 \approx \int \mathrm{d}x\mathrm{d}\tau\,\bar{\psi}_R\left(\partial_\tau - iv_F\partial_x\right)\psi_R + \bar{\psi}_L\left(\partial_\tau + iv_F\partial_x\right)\psi_L \tag{3.35}$$

$$= \sum_{\eta=R,L}\int \mathrm{d}x\mathrm{d}\tau\,\bar{\psi}_\eta\left(\partial_\tau - i\eta v_F\partial_x\right)\psi_\eta. \tag{3.36}$$

Thus, we see that the action splits into two independant chiral sectors. If we can safely neglect curvature corrections (these would become important e.g. at the bottom of the band) Eq. (3.36) is a universal model for the low energy excitations of one-dimensional electrons. If we define $\gamma_0 = \sigma_x$, $\gamma_1 = \sigma_y$, the two-spinors $\bar{\Psi} = \left(\bar{\psi}_R, \bar{\psi}_L\right)\sigma_x$, $\Psi = (\psi_R, \psi_L)^T$ and two dimensional space time $\mathbf{x} = (x, \tau)$ the action of a free one-dimensional fermion is identified with that of a (1+1) dimensional Dirac particle

$$S_0 = \int \mathrm{d}^2x\, \bar{\Psi}\gamma_\mu \partial^\mu \Psi. \tag{3.37}$$

The free propagators are readily calculated:

$$\langle \psi_R(\mathbf{x})\bar{\psi}_R(\mathbf{x}')\rangle = \frac{1}{2\pi}\frac{1}{z-z'}, \qquad \langle \psi_L(\mathbf{x})\bar{\psi}_L(\mathbf{x}')\rangle = \frac{1}{2\pi}\frac{1}{\bar{z}-\bar{z}'}. \tag{3.38}$$

We can also include external fields by the standard minimal subtitution $\partial_\mu \to \partial_\mu - ieA_\mu$ [7] to get

$$S_0 = \int \mathrm{d}^2x\, \bar{\Psi}\gamma^\mu \left(\partial_\mu - ieA_\mu\right)\Psi. \tag{3.39}$$

This enables us to calculate the current density $j = j_1$ and charge density $\rho = ij_0$ by variation of the action,

$$j_\mu = \frac{\delta S[A]}{\delta A_\mu}, \tag{3.40}$$

which yields:

$$\rho = e\left(J_R + J_L\right), \tag{3.41}$$
$$J = e\left(J_R - J_L\right). \tag{3.42}$$

Here we defined the chiral currents $J_R = \bar{\psi}_R\psi_R$, $J_L = \bar{\psi}_L\psi_L$. At this point we want to add the following remarks:

- One-dimensional electrons are primary fields and we can read off their conformal dimension from the propagators (3.38):

$$\psi_R : (h, \bar{h}) = \left(\frac{1}{2}, 0\right), \qquad \psi_L : (h, \bar{h}) = \left(0, \frac{1}{2}\right). \tag{3.43}$$

- In a one-dimensional Fermi system with uniform velocity v_F, the current density is proportional to the number of right movers minus that of left movers. Thus it is fundamentally different than higher dimensional systems where the current density is proportional to momentum.[8] In other words a process that conserves momentum can still affect transport properties if it changes the number of right (left) movers. This fact will become important later.

3.3. Peculiarities of one dimension

After introducing some formal concepts let us now turn to a more physical explanation of one-dimensional systems. Before explaining why one dimension is special, let us first review what makes higher dimensions normal. The effect of interactions in higher dimensions is described by the Fermi liquid theory.

[7] Remember $c \equiv 1$ and e is charge. Furthermore, in the imaginary time formalism $A_0 = i\phi$.
[8] A notable exception to this is graphene, see e.g. Ref. [31].

Free fermions obey Fermi-Dirac statistics and their groundstate at zero temperature is the filled Fermi sea. Excitations with respect to this groundstate are electrons defined by their momentum, spin and charge $(\mathbf{k}, \sigma, e_0)$. Since they are eigenstates of the Hamiltonian they have infinite lifetime. Landau's hypothesis was, that if we adiabatically switch on interactions, the fundamental quantum numbers $(\mathbf{k}, \sigma, e_0)$ and the statistics of the particles are left unchanged. However, their dynamical properties such as mass, velocity etc. get renormalized by the effect of interactions. The excitations are thus not the fundamental electrons anymore but are called quasiparticles. Heuristically, these are electrons dressed by surrounding density fluctuations.

Formally, these properties can be seen in the single particle propagator $G(\mathbf{k}, \omega)$ and the spectral function $A(\mathbf{k}, \omega)$.
The poles of the propagator contain information about the quasiparticle spectrum of an interacting system. The poles' real part specifies the quasiparticle energy while the imaginary part contains information about the lifetime. On the other hand the spectral function $A(\mathbf{k}, \omega)$ can be interpreted as the probability of finding a quasiparticle in the state $(\mathbf{k}, \omega)$. For free electrons at zero temperature we have

$$G(\mathbf{k}, \omega) = \frac{1}{\omega - \xi_{\mathbf{k}} + i0\,\mathrm{sign}(|\mathbf{k}| - |\mathbf{k}_F|)}, \qquad A(\mathbf{k}, \omega) = \delta(\epsilon - \xi_{\mathbf{k}}). \tag{3.44}$$

If one includes interactions the self energy Σ of quasiparticles becomes finite. This renormalizes their properties and broadens the spectral weight to a Lorentzian form. However, the quantum numbers of exitations are left unchanged, i.e. we get

$$G(\mathbf{k}, \omega) = \frac{Z_{\mathbf{k}}}{\omega - \tilde{\xi}_{\mathbf{k}} - \frac{i}{2}\Gamma(\mathbf{k}, \tilde{\xi}_{\mathbf{k}})}, \qquad A(\mathbf{k}, \omega) = \frac{Z_{\mathbf{k}}\Gamma(\mathbf{k}, \tilde{\xi}_{\mathbf{k}})}{\left(\omega - \tilde{\xi}_{\mathbf{k}}\right)^2 + 4\Gamma(\mathbf{k}, \tilde{\xi}_{\mathbf{k}})}, \tag{3.45}$$

where $Z \in [0, 1]$ is the quasiparticle weight determining the fraction of electrons that are in the quasiparticle state, $\tilde{\xi}_{\mathbf{k}}$ is the quasiparticle dispersion relation and $\Gamma(\mathbf{k}, \tilde{\xi}_{\mathbf{k}})$ is the rate of decay of the state $(\mathbf{k}, \tilde{\xi}_{\mathbf{k}})$,

$$\tilde{\xi}_{\mathbf{k}} = \xi_{\mathbf{k}} + \mathfrak{Re}\,\Sigma(\mathbf{k}, \tilde{\xi}_{\mathbf{k}}), \tag{3.46}$$

$$Z_{\mathbf{k}}^{-1} = 1 - \partial_\epsilon\, \mathfrak{Re}\,\Sigma(\mathbf{k}, \omega)|_{\omega = \tilde{\xi}_{\mathbf{k}}}, \tag{3.47}$$

$$\Gamma(\mathbf{k}, \tilde{\xi}_{\mathbf{k}}) = 2Z_{\mathbf{k}}\, \mathfrak{Im}\,\Sigma(\mathbf{k}, \omega). \tag{3.48}$$

The results are summarized in Fig. 3.2.

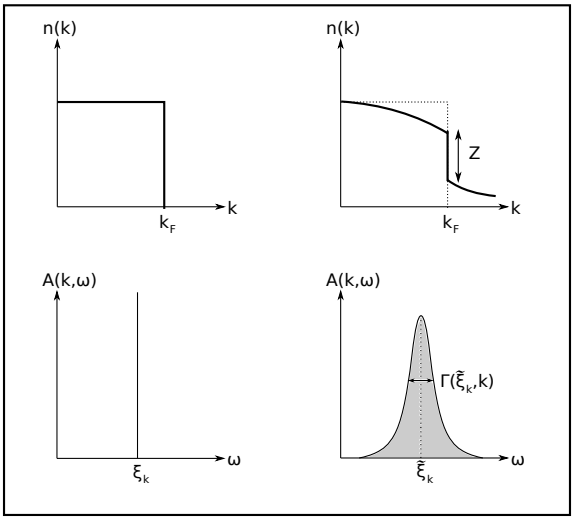

Figure 3.2.: Distribution function n(k) and spectral function $A(\mathbf{k}, \omega)$ for free fermions (left) and quasiparticles (right).

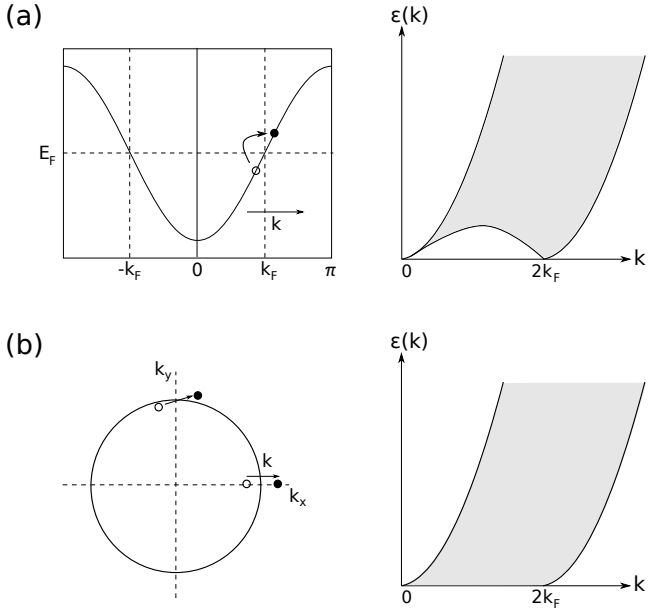

Figure 3.3.: Particle hole spectrum in 1d (a) and in 2d (b).

Quasiparticles are only well-defined if their inverse lifetime is much smaller than any characteristic energy scale of the problem. Fortunately, in dimensions higher than one this is usually the case and excitations are well-defined in the vicinity of the Fermi surface.

The remarkable feature of Fermi liquid theory is that it does not rely on weak coupling but holds for arbitrary interaction strenghts in the absence of symmetry breaking phase transitions. It is thus all the more suprising that it fails to describe interaction effects in one dimension.
Indeed, if we calculate the electronic self energy in 1D we find that the inverse quasiparticle weight is logarithmically diverging with the energy cutoff Λ. Our statement is thus: in one dimension there exist no coherent single particle exitations but only collective modes. To see this, consider a cartoon picture of a one-dimensional wire with electrons on it. If we insert an additional electron into the wire it will push away the neighboring electrons. Thus it creates a density fluctuation which is a collective excitation of all the electrons in the wire.

On a more formal level, consider Fig. 3.3 where we plot the energy of particle hole pairs created out of the Fermi sea against their momenta (both measured w.r.t. the groundstate). Patched regions show the points in parameter space where electron hole excitations are allowed. Due to the linear dispersion near the Fermi points, pairs have a narrow dispersion near $k = 0$. That means a pair has a well-defined dispersion relation and can propagate coherently. In other words, because particles and holes have nearly the same group velocity [9] they propagate together and a weakly attractive interaction can bind them into a coherently propagating entity: a new particle. In higher dimensions these arguments don't apply since the Fermi surface is not disjoint. Therefore, pairs can have arbitrary energy starting from zero and interactions have a harder time forming coherent excitations. The fact that particle hole excitations have bosonic character and that the low energy physics is dominated by them is the basis for the bosonization procedure which will be discussed in the next section.

[9] Due to the linearized spectrum $v = \partial_k E(k) = \text{const.}$.

3.4. Operator bosonization

In order to calculate correlation functions it was useful to work in the action formalism. However, from now on we will switch to the operator representation because it will be used throughout most of this thesis. If we define the correspondence

$$\hat{\psi}_R(z) = \frac{1}{\sqrt{2\pi}} e^{-i\sqrt{4\pi}\hat{\phi}_R(z)}, \qquad \hat{\psi}_L(\bar{z}) = \frac{1}{\sqrt{2\pi}} e^{i\sqrt{4\pi}\hat{\phi}_L(\bar{z})}, \tag{3.49}$$

the correlation functions between the fermionic model and bosonic exponentials will be identical. In fact it turns out that the entire spectrum of the two models is identical and Eq. (3.49) is an exact mapping between them. This remains true even in the presence of interactions. We are not going to prove this equivalence here but refer to the literature given at the beginning of this chapter.

While the bosonization formulas (3.49) are the basis of fermion-to-boson translation, their naive implementation often leads to errors. Take for example the current of right movers $\hat{J}_R(x) = \hat{\psi}_R^\dagger(x)\hat{\psi}_R(x)$. It is formally infinite because there is an infinite number of holes occupying the states at point x. Thus to regularize infinities like this we have to normal order the operator. Equivalently, we can implement the procedure of *point splitting* by defining

$$\hat{J}(z) = \lim_{\epsilon\to 0}\left[\hat{\psi}^\dagger(z+\epsilon)\hat{\psi}(z) - \langle\hat{\psi}^\dagger(z+\epsilon)\hat{\psi}(z)\rangle\right], \tag{3.50}$$

and similar for the left moving current. This yields

$$\hat{J}_R = \frac{i}{\sqrt{\pi}}\partial_z\hat{\varphi}, \qquad \hat{J}_L = -\frac{i}{\sqrt{\pi}}\partial_{\bar{z}}\hat{\varphi}. \tag{3.51}$$

The above bosonization neglects some further subleties:

- The fermionic operators have dimension $L^{-\frac{1}{2}}$, thus we have to add a UV cutoff a_c into the normalization to regularize formally divergent integrals. While a_c is equal to the lattice spacing a_0 in non-interacting models, they generally are non-trivially related in interacting ones.
- To ensure fermionic anticommutation relations one has to introduce Hermitian operators called Klein factors $\hat{\kappa}_j$ that connect the ground state of different charge sectors and obey the Clifford algebra:

$$\{\hat{\kappa}_i, \hat{\kappa}_j\} = \{\hat{\bar{\kappa}}_i, \hat{\bar{\kappa}}_j\} = \delta_{ij}, \qquad \{\hat{\kappa}_i, \hat{\bar{\kappa}}_j\} = 0. \tag{3.52}$$

  They can often be neglected and anticommutativity is ensured by demanding

$$\left[\hat{\phi}_R(x), \hat{\phi}_R(x')\right] = -\frac{i}{4}\mathrm{sign}(x-x'), \tag{3.53}$$

$$\left[\hat{\phi}_L(x), \hat{\phi}_L(x')\right] = \frac{i}{4}\mathrm{sign}(x-x'). \tag{3.54}$$

  However, they are necessary if there is more than one species of fermions, e.g. in the presence of spin.

The bosonization formulas are therefore:

$$\hat{\psi}_{R,j}(z) = \frac{\hat{\kappa}_j}{\sqrt{2\pi a_c}} e^{-i\sqrt{4\pi}\hat{\phi}_{R,j}(z)}, \qquad \hat{\psi}_{L,j}(\bar{z}) = \frac{\hat{\bar{\kappa}}_j}{\sqrt{2\pi a_c}} e^{i\sqrt{4\pi}\hat{\phi}_{L,j}(\bar{z})}. \tag{3.55}$$

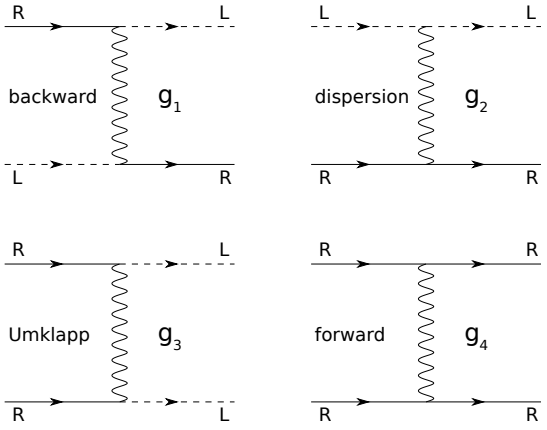

Figure 3.4.: Different interaction processes in a LL.

3.5. Interacting electrons

For physical electrons we must add a spin index $\sigma =\uparrow,\downarrow$. Furthermore, we need to introduce two boson operators $\hat{\varphi}_\uparrow$ and $\hat{\varphi}_\downarrow$. These may naturally be combined into spin and charge components:

$$\hat{\varphi}_c = \frac{1}{\sqrt{2}}\left(\hat{\varphi}_\uparrow + \hat{\varphi}_\downarrow\right), \qquad \hat{\varphi}_s = \frac{1}{\sqrt{2}}\left(\hat{\varphi}_\uparrow - \hat{\varphi}_\downarrow\right), \tag{3.56}$$

and the same for the chiral bosons $\hat{\phi}_{R(L),c(s)}$.

In the low energy limit the effect of interactions is limited to the vicinity of the Fermi points and processes fall into 4 categories:[10]

1. backward scattering

$$\hat{\mathcal{H}}_1 = v_F g_1 \sum_\sigma \hat{\psi}^\dagger_{R,\sigma}\hat{\psi}_{L,\sigma}\hat{\psi}^\dagger_{L,-\sigma}\hat{\psi}_{R,-\sigma}. \tag{3.57}$$

2. dispersive scattering

$$\hat{\mathcal{H}}_{2,c} = v_F g_{2,c}\left(\hat{J}_{R,\uparrow} + \hat{J}_{R,\downarrow}\right)\left(\hat{J}_{L,\uparrow} + \hat{J}_{L,\downarrow}\right), \tag{3.58}$$

$$\hat{\mathcal{H}}_{2,s} = v_F g_{2,s}\left(\hat{J}_{R,\uparrow} - \hat{J}_{R,\downarrow}\right)\left(\hat{J}_{L,\uparrow} - \hat{J}_{L,\downarrow}\right). \tag{3.59}$$

3. Umklapp scattering (only allowed at half filling, because of crystal momentum conservation)[11]

$$\hat{\mathcal{H}}_3 = \frac{1}{2} v_F g_3 \sum_\sigma \hat{\psi}^\dagger_{R,\sigma}\hat{\psi}^\dagger_{R,-\sigma}\hat{\psi}_{L,\sigma}\hat{\psi}_{L,-\sigma} + h.c.. \tag{3.60}$$

[10] You could ask yourself if there isn't also a fifth process that changes only one chirality e.g. $\psi^\dagger_R\psi^\dagger_R\psi_L\psi_R$. However, this term only conserves momentum at full filling $k_F = 2\pi$ where the system is an insulator anyway and at $k_F = 0$ which is unphysical because the linearisation procedure is ill defined at the bottom of the band.

[11] To see this consider the operator in momentum space e.g. $\bar{\psi}_{R,k}\bar{\psi}_{R,p}\psi_{L,k+q}\psi_{L,p-q}$ then we must have $k, p \sim k_F$, $q \sim -2k_F$. In order for all the operators to be near the Fermi points we must add a reciprocal lattice vector $Q = 2\pi/a_0$. In the case of half filling, $k_F = \pi/(2a_0)$, we have $Q = 4k_F$ and thus $p-q-Q$ is close to the left Fermi point as needed.

4. forward scattering

$$\hat{\mathcal{H}}_{4,c} = \frac{1}{2} v_F g_{4,c} \left[\left(\hat{J}_{R,\uparrow} + \hat{J}_{R,\downarrow} \right)^2 + \left(\hat{J}_{L,\uparrow} + \hat{J}_{L,\downarrow} \right)^2 \right], \tag{3.61}$$

$$\hat{\mathcal{H}}_{4,s} = \frac{1}{2} v_F g_{4,s} \left[\left(\hat{J}_{R,\uparrow} - \hat{J}_{R,\downarrow} \right)^2 + \left(\hat{J}_{L,\uparrow} - \hat{J}_{L,\downarrow} \right)^2 \right]. \tag{3.62}$$

The interaction vertices for all processes are depicted in Fig. 3.4.

3.5.1. Tomonaga-Luttinger liquid

Let us neglect scattering processes across the Fermi points (g_1, g_3) for now. What we are left with is called the *Tomonaga-Luttinger model* which can be solved exactly by bosonization. In the bosonized Hamiltonian spin and charge degrees of freedom decouple

$$\hat{\mathcal{H}}_{\mathrm{T.L}} = \hat{\mathcal{H}}_c + \hat{\mathcal{H}}_s, \tag{3.63}$$

$$\hat{\mathcal{H}}_j = \hat{\mathcal{H}}_{0,j} + \hat{\mathcal{H}}_{2,j} + \hat{\mathcal{H}}_{4,j}. \tag{3.64}$$

Here we defined

$$\hat{\mathcal{H}}_{0,j} = \frac{v}{2} \left[\hat{\Pi}_j^2 + \left(\partial_x \hat{\varphi}_j \right)^2 \right], \tag{3.65}$$

$$\hat{\mathcal{H}}_{2,j} = -\frac{v g_{2,j}}{2\pi} \left[\hat{\Pi}_j^2 - \left(\partial_x \hat{\varphi}_j \right)^2 \right], \tag{3.66}$$

$$\hat{\mathcal{H}}_{4,j} = \frac{v g_{4,j}}{2\pi} \left[\hat{\Pi}_j^2 + \left(\partial_x \hat{\varphi}_j \right)^2 \right], \tag{3.67}$$

and j=c,s. Combining the expressions we get

$$\hat{\mathcal{H}}_j = \frac{1}{2} u_j \left[K_j \hat{\Pi}_j^2 + \frac{1}{K_j} \left(\partial_x \hat{\varphi}_j \right)^2 \right], \tag{3.68}$$

where

$$K_j = \sqrt{\frac{\pi - g_{2,j} + g_{4,j}}{\pi + g_{2,j} + g_{4,j}}}, \qquad u_j = v \sqrt{\left(1 + \frac{g_{4,j}}{\pi} \right)^2 - \left(\frac{g_{2,j}}{\pi} \right)}. \tag{3.69}$$

The parameter u_j is the renormalized Fermi velocity (plasmon velocity) and K_j is called the Luttinger liquid parameter and is a measure for the interaction strength. For repulsive interaction $g_{2,j}, g_{4,j} > 0$ the Luttinger parameter is $K_c < 1$, for attractive interactions $g_{2,j}, g_{4,j} < 0$ we have $K_c > 1$ and for a non-interacting system $K = 1$. The corresponding action (for one species) is:

$$S = \frac{u}{2} \int \mathrm{d}x \mathrm{d}\tau \left[K \left(\partial_x \theta \right)^2 + K^{-1} \left(\partial_\tau \varphi \right)^2 + 2 i u^{-1} \partial_\tau \varphi \partial_x \theta \right]. \tag{3.70}$$

If we integrate out φ or θ fields, respectively, we arrive at the two equivalent descriptions of the LL:

$$S[\varphi] = \frac{u}{2K} \int \mathrm{d}x \mathrm{d}\tau \left[\left(\partial_x \varphi \right)^2 + \frac{1}{u^2} \left(\partial_\tau \varphi \right)^2 \right], \tag{3.71}$$

$$S[\theta] = \frac{uK}{2} \int \mathrm{d}x \mathrm{d}\tau \left[\left(\partial_x \theta \right)^2 + \frac{1}{u^2} \left(\partial_\tau \theta \right)^2 \right]. \tag{3.72}$$

This can be brought back into canonical form by the transformation $\varphi_j \to \sqrt{K_j} \varphi_j$, $\theta_j \to 1/\sqrt{K_j} \theta_j$. Therefore, we see that K_j just renormalizes the fields.

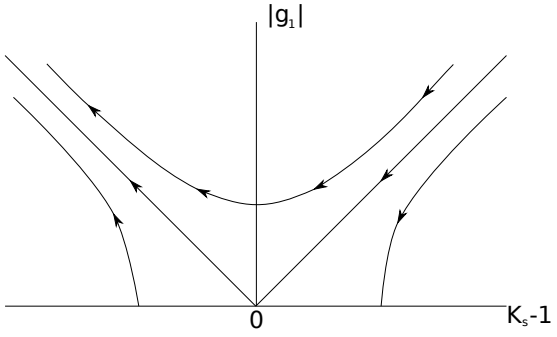

Figure 3.5.: RG flow of the sine-Gordon model.

Also note that the above action describes collective excitations (the Euler Lagrange equation of motion describes a wave of velocity u). These are called charge density wave (CDW) or spin density wave (SDW) in the charge or spin sector, respectively.

3.5.2. Spin or charge gap and the sine-Gordon model

If we take into account $\hat{\mathcal{H}}_3, \hat{\mathcal{H}}_1$ the model is not exactly solvable anymore, but we can perform a weak coupling RG analysis. Let us first consider the g_1 term. Combined with the bosonized spin Hamiltonian we get one of the most studied models in field theory, the sine-Gordon model:

$$\hat{\mathcal{H}}_{\mathrm{SG}} = \frac{u}{2}\left[\hat{\Pi}_s^2 + (\partial_x \hat{\varphi}_s)^2\right] + \frac{u g_1}{2\pi^2}\cos\left(\sqrt{8\pi K_s}\hat{\varphi}_s\right) \tag{3.73}$$

(here we used the rescaled fields so that the free part has the standard form (3.17)). The scaling dimension of the cosine operator is $\Delta = 2K_s$ (see 3.31). Naively we would expect that it is relevant for $K_s < 1$ and irrelevant for $K_s > 1$. However, K_s is subject to the RG flow itself and thus we have to study the coupled RG equations

$$\frac{dK_s}{dl} = -\frac{1}{2\pi^2}K_s^2 g_1^2, \qquad \frac{dg_1}{dl} = -2g_1(K_s - 1). \tag{3.74}$$

These were first obtained by Kosterlitz and Thouless [32]. The corresponding flow diagram is illustrated in Fig. 3.5. At weak coupling the system flows to strong coupling if $|g_1| > 2\pi(K_s - 1)$ (g_1 is *marginally relevant*) while it flows to $g_1 = 0$ if $|g_1| < 2\pi(K_s - 1)$ (g_1 is *marginally irrelevant*). Therefore, at $K_s < 1$ even a infinitesimal g_1 can open a spin gap. Exactly the same analysis can be performed for the g_3 term (we just have to substitute $g_1 \to g_3$ and $K_s \to K_c$ in all the expressions). Now what does the creation of a gap mean in more physical terms? The cosine term represents a cosine-potential and would like to lock the field φ in one of its minima. On the other hand the kinetic term is minimized by large spatial fluctuations of the field φ.[12] Thus if the cosine term is irrelevant the particles motion is undisturbed while in the case where the cosine term is relevant it is locked at one of the minima of the potential. The excitations of the system are then tunneling processes between minima, which need a minimal energy of the order of the gap, called solitons. In the case of spin gap they are called spinons and for a charge gap holons. These excitations correspond to free massive particles.

[12] Since $\hat{\Pi}$ and $\hat{\varphi}$ are conjugate operators there is an uncertanty relation between them. In the extreme case where $\hat{\varphi}$ is constant in space, $\hat{\Pi}$ would be infinite. Therefore, the kinetic term is minimized by spatial fluctuations.

3.6. Disorder in Luttinger liquids

In Sec. 3.5 we discussed the physics of interacting one-dimensional electrons. We saw that, unlike in a Fermi liquid, the fundamental excitations of the system are charge (and spin) density waves - collective excitations describing wave like propagation of charge and spin degrees of freedom, respectively. Now let us discuss what happens if we introduce localized impurities into the system. This problem is of considerable practical relevance. Every physical realisation of one-dimensional systems - semiconducting quantum wires, nanotubes, quantum Hall edges, etc. - generally contain imperfections such as dislocations, vacancies or charged impurities. In the following we will only consider static impurities without intrinsic degrees of freedom, that could be excited in a scattering event. Hence, scattering off impurities is always elastic.
For a review of disorder effects in higher dimension see the textbooks [27, 33] and specifically for one-dimensional systems the review [34].

What are the general physical consequences of impurities in a system?

1. Translation symmetry is broken and therefore (crystal-) momentum is not conserved anymore.
2. Non-interacting electrons acquire a finite self energy and consequently a lifetime called the transport scattering time τ_{tr}.
3. Besides τ_{tr} there is another relevant time scale - the coherence time τ_ϕ. This is the time scale on which the phase of electrons is randomized due to electron-electron scattering or phonons. If $\tau_\phi/\tau_{tr} \gtrsim 1$ interference effects of coherent electrons may cause localization.[13]

Additionally, unlike in higher dimensions a charge (spin) degree of freedom propagating down a one-dimensional channel will inevitably hit the impurities. Thus we expect impurities to have a much stronger impact on transport characteristics than in higher dimensions. There is also one further not so obvious mechanism behind this strong impact.
Imagine a wavepacket of characteristic momentum k_F colliding with an impurity at $x = 0$. The resulting wave amplitude to the left of the impurity $\psi(x) \sim e^{ik_F x} + re^{-ik_F x}$ will be a linear superposition of the incoming amplitude and the reflected amplitude, where r is the reflection coefficient. Therefore, the electronic density profile is given by $\rho(x) = |\psi(x)|^2 \sim 1 + |r|^2 + 2\,\mathfrak{Re}\, re^{-2ik_F x}$. The oscillatory contribution is known as *Friedel oscillation*. In one dimension these decay rather slowly ($\sim |x|^{-1}$). The key point is, that electron waves approaching the impurity will not only notice the impurity potential but also the additional scattering potential off Friedel oscillations. Moreover the additional scattering potential creates secondary Friedel oscillations and so on. Therefore, even a weak impurity potential may be amplified by the effect of Friedel oscillations. This is most impressively shown in the Kane-Fisher problem (see Sec. 3.6.3).
Finally, as was first shown in a seminal work by the "gang of four" [35] arbitrarily weak disorder in a non-interacting system leads to localization of all states in $d \leq 2$ dimensions. For one dimension this was known even earlier [36]. Therefore, the DC conductivity is zero for all temperatures $\sigma_{DC}(T) = 0$. On the other hand, the AC conductivity is strongly suppressed for weak external frequencies Ω, i.e. $\Omega\tau_{tr} \ll 1$ and has the normal Drude form with additional weak localization corrections for $\Omega\tau_{tr} \gg 1$. At $\Omega\tau_{tr} \sim 1$ the ballistic regime directly crosses to the strongly localized regime and the diffusive regime is absent.

[13] Localization describes the effect that electrons on time reversed paths interfere constructively. Therefore, the probability for an electron to return to its starting point is enhanced and it may become stuck in one region of space. This is called weak localization. If such processes accumulate an electron can become strongly localized, i.e. the electronic wavefunction decays rapidly away from its position on a characteristic length scale ξ (localization length). In this case the system undergoes a phase transition from metal to insulator where the order parameter is the conductance G (Anderson transition).

Now that we know what happens in clean systems (LL) and in disordered but non-interacting systems (Anderson localization) let us end this chapter by some case studies on the combined effect of disorder and interactions in low dimensional systems. However, first we have to introduce further formal concepts.

3.6.1. Gaussian model for disorder

In a specific system disorder is described by the continous potential U(x). However, the field U(x) will differ from one system to another since the microscopic arrangement of impurities is always different. Therefore, we will argue that U is a random variable that is described by a probability distribution $\mathcal{P}[U]$, i.e. $\mathcal{P}[U_0]\mathcal{D}[U_0]$ is the probability that the specific potential U_0 is realized. $\mathcal{P}[U]$ is often chosen to be Gaussian, because due to the central limit theorem any distribution will become Gaussian if one considers a large ensemble of systems. Therefore, we choose

$$\mathcal{P}[U] = \exp\left[-\frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\, U(x)K^{-1}(x-x')U(x')\right] \tag{3.75}$$

where K describes the spatial correlation profile of impurities. If we assume that physical observables vary on much larger length scales than the mean distance between impurities, the behavior of an observable will always be affected by a whole ensemble of disorder realizations in an area where it itself is approximately constant. Therefore, we have to average the observables over the distribution $\mathcal{P}[U]$. To this end we define the mean value and the variance of U,

$$\langle U(x)\rangle_{dis} = 0, \tag{3.76}$$
$$\langle U(x)U(x')\rangle_{dis} = K(x-x'). \tag{3.77}$$

Here the average is with respect to $\mathcal{P}[U]$,

$$\langle\cdots\rangle_{dis} \equiv \int \mathcal{D}U\, \mathcal{P}[U]\,(\cdots) \tag{3.78}$$

and K is often chosen as $K(x-x') = D_{dis}\delta(x-x')$, where D_{dis} is the disorder strength. This just expresses the fact that we assume the correlation length of disorder to be much smaller than any other characteristic length scale of electrons.

In the Matsubara technique the average over thermal and quantum fluctuations of the field ϕ of an observable O is

$$\langle O\rangle = \frac{\int \mathcal{D}\left[\phi\right]\, e^{-S[\phi]}O\left[\phi\right]}{\int \mathcal{D}\left[\phi\right]\, e^{-S[\phi]}}, \tag{3.79}$$

where the action is $S = S_0 + \int \mathrm{d}x\mathrm{d}\tau\, U(x)F\left[\phi\right]$ and F is an analytic functional of the field ϕ which describes the coupling to the disorder potential. Next we have to average over all random potentials,

$$\langle\langle O\rangle\rangle_{dis} = \frac{\int \mathcal{D}\left[U\right]\, \mathcal{P}\left[U\right]\langle O\rangle}{\int \mathcal{D}\left[U\right]\, \mathcal{P}\left[U\right]}. \tag{3.80}$$

If it were not for the denominator in (3.79), the averaging over disorder would be straightforward. To date, there exist three ways to prevent the denominator and implement the disorder average: the replica trick [27], the Keldysh technique [17] and the supersymmetry approach [37]. However, here we will discuss none of them but just refer to the literature.

3.6.2. Action of one-dimensional disordered systems

The disorder term in the action is

$$S_{\text{dis}} = \int \mathrm{d}x\mathrm{d}\tau\, U(x)\bar{\psi}\psi = \frac{1}{L}\sum_{k,q}\int \mathrm{d}\tau\, U_q \bar{\psi}_{k+q}\psi_k. \tag{3.81}$$

If the impurity strength D_{dis} is much smaller than the Fermi energy, disorder only produces effects close to the Fermi points. Consequently, impurities can only scatter forward ($q \sim 0$) or backward ($q \sim \pm 2k_F$).

Let us define the corresponding disorder amplitudes

$$\mathcal{U}_f(x) = \frac{1}{L}\sum_{q\sim 0} U_q e^{iqx}, \tag{3.82}$$

$$\mathcal{U}_b(x) = \frac{1}{L}\sum_{q\sim 0} U_{q-2k_F} e^{iqx}. \tag{3.83}$$

While $\mathcal{U}_f$ is real,[14] $\mathcal{U}_b$ is in general complex. The correlation functions for $\mathcal{U}_b$ read

$$\langle \mathcal{U}_b(x)\mathcal{U}_b(x')\rangle = D_b\delta(x-x'). \tag{3.84}$$

Furthermore, we can decompose the fermionic field ψ as in (3.33). The resulting disorder action is:

$$S_{\text{dis}} = \int \mathrm{d}x\mathrm{d}\tau\, \left[\mathcal{U}_f(x)\left(\bar{\psi}_R\psi_R + \bar{\psi}_L\psi_L\right) + \mathcal{U}_b(x)\bar{\psi}_R\psi_L + \mathcal{U}_b^*(x)\bar{\psi}_L\psi_R\right]. \tag{3.85}$$

If we bosonize the model the resulting action including both interactions (g_2, g_4) and disorder is

$$\begin{aligned} S =& \frac{u}{2K}\int \mathrm{d}x\mathrm{d}\tau\, \left[(\partial_x\varphi)^2 + \frac{1}{u^2}(\partial_\tau\varphi)^2\right] - \int \mathrm{d}x\mathrm{d}\tau\, \frac{\mathcal{U}_f}{\sqrt{\pi}}\partial_x\varphi \\ &+ \frac{1}{2\pi a_c}\int \mathrm{d}x\mathrm{d}\tau\, \mathcal{U}_b^*(x)e^{i\sqrt{4\pi}\varphi(x)} + h.c.. \end{aligned} \tag{3.86}$$

An important fact to notice is, that forward scattering can be completely removed by the transformation $\varphi \to \varphi + \int^x \mathrm{d}y\, \mathcal{U}_f(y)K/(u\sqrt{\pi})$[15]. This is just a formal manifestation of the argument that only processes that change the chirality of electrons affect the current. Our final result for the action is thus:

$$S = \frac{u}{2\pi K}\int \mathrm{d}x\mathrm{d}\tau\, \left[(\partial_x\varphi)^2 + \frac{1}{u^2}(\partial_\tau\varphi)^2\right] + \frac{1}{2\pi a_c}\int \mathrm{d}x\mathrm{d}\tau\, \mathcal{U}_b^*(x)e^{2i\varphi(x)} + h.c.. \tag{3.87}$$

Here we also scaled $\varphi \to 1/\sqrt{\pi}\varphi$ to make the notation consistent with [25].

3.6.3. Single impurity in a one-dimensional wire (Kane-Fisher problem)

The Kane-Fisher problem considers a single impurity in an one-dimensional wire. For simplicity we restrict ourselves to spinless fermions. It turns out, that at $T = 0$ depending on the interaction strength even a single impurity can drive the system to an insulator for repulsive interaction ($K < 1$), while it remains a perfect conductor for attractive interaction ($K > 1$) [38].

[14] If U(x) is real, $U_q = U_{-q}$.

[15] The attentive reader may argue that this transformation also changes the backscattering part because $\mathcal{U}_b^* \to \mathcal{U}_b^* e^{2i\int^x \mathrm{d}y\, \mathcal{U}_f(y)K/u}$. However, since $\mathcal{U}_b$ was a random variable to begin with and its Gaussian statistics are not affected by the additional phase the action is unchanged by this.

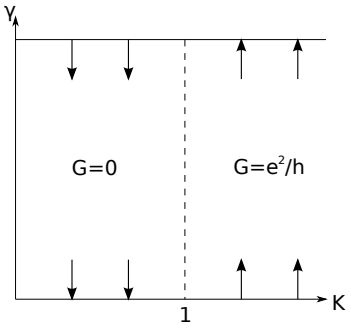

Figure 3.6.: Behavior of a single impurity in a one-dimensional wire.

At finite T, conductance is shown to vanish as a power law where the exponent is a function of K. To show this, consider an interacting electron system in a one-dimensional wire with an impurity at $x = 0$. The corresponding action is given by (3.87) where $\mathcal{U}_b(x) = \mathcal{U}_b\delta(x)$ is now a local term. Further it can be chosen to be real, since any phase that may be associated with $\mathcal{U}_b$ can be removed by a global gauge transformation of the fermionic fields, i.e. suppose

$$\mathcal{U}_b = |\mathcal{U}_b|\, e^{2i\chi}, \tag{3.88}$$

$$\text{then scale: } \bar{\psi}_R \to e^{-i\chi}\bar{\psi}_R, \qquad \psi_L \to e^{-i\chi}\psi_L. \tag{3.89}$$

Thus the action reads

$$S = \frac{u}{2\pi K}\int \mathrm{d}x\mathrm{d}\tau\, \left[(\partial_x\varphi)^2 + \frac{1}{u^2}\,(\partial_\tau\varphi)^2\right] + \gamma\int \mathrm{d}\tau\, \cos(2\varphi(x=0,\tau)), \tag{3.90}$$

where $\gamma = \mathcal{U}_b(\pi a_c)^{-1}$. The important fact is that the action is quadratic in all fields $\varphi(x,\tau)$ except for $\varphi(x=0,\tau) \equiv \varphi(\tau)$. If we therefore integrate out all quadratic degrees of freedom we end up with an effective action for $x = 0$ that is local in space but highly nonlocal in (imaginary) time,

$$S_{\text{eff}}[\varphi(\tau)] = \frac{1}{\pi T K}\sum_n \varphi_n\, |\omega_n|\, \varphi_{-n} + \gamma\int \mathrm{d}\tau\, \cos(2\varphi(\tau)), \tag{3.91}$$

where $\varphi_n = 1/\sqrt{\beta}\int_0^\beta \mathrm{d}\tau\, \psi(\tau)e^{i\omega_n\tau}$ is the Fourier transform of $\varphi(\tau)$ to Matsubara frequencies. We can now study the behavior of the system by performing a perturbative one loop RG in the small parameter γ to get the beta function

$$\frac{d\gamma}{d\ln(b)} = \gamma\,(1-K). \tag{3.92}$$

Thus, as discussed in the beginning, the potential term becomes relevant for $K < 1$ but is irrelevant for $K > 1$. That means that for attractive interactions the effect of the barrier dissapears completely while it becomes very large for repulsive interactions. It can be shown that this remains true even for a large impurity potential, where our perturbative RG loses its meaning. The corresponding phase diagram is shown in Fig. 3.6.

3.6.4. Case of many impurities (Giamarchi-Schulz RG)

Let us now look at the case of many impurities following Ref. [34]. In order to perform the disorder average we introduce replicas φ_n. Using the Gaussian nature of the disorder distribution we can write $\langle e^{S_{dis}}\rangle_{dis} = e^{-\frac{1}{2}\langle S_{dis}^2\rangle_{dis}} \equiv e^{-\frac{1}{2}S_D}$ to arive at the disorder averaged

action:

$$S = \sum_n S_{LL}\left[\varphi_n\right] + \sum_{n,m} S_D\left[\varphi_n, \varphi_m\right], \tag{3.93}$$

$$S_{LL} = \frac{1}{uK}\int \frac{\mathrm{d}x\mathrm{d}\tau}{2\pi}\left\{\left[\partial_\tau \varphi_n(x,\tau)\right]^2 + u^2\left[\partial_x \varphi_n\right]^2\right\}, \tag{3.94}$$

$$S_D = -D_b \int \frac{\mathrm{d}x\mathrm{d}\tau\mathrm{d}\tau'}{\left(2\pi a_c\right)^2}\cos\left\{2\left[\varphi_n(x,\tau) - \varphi_m(x,\tau')\right]\right\}. \tag{3.95}$$

Giamarchi and Schulz derived the coupled renormalization equations for the coupling constants K,u and D_b upon rescaling the UV cutoff $a_c \to L = a_c e^l$. This analysis is done to first order in the dimensionless strength of disorder,

$$D = \frac{2D_b a_c}{\pi u^2}. \tag{3.96}$$

To this order one can drop the replica indices. Furthermore, they singled out the contribution of close times $u\left|\tau - \tau'\right| \leq a_c$ in the double time integral. Back in the fermionic language this becomes

$$\begin{aligned} S_D = &-D_b \int_{u|\tau-\tau'|>a_c} \mathrm{d}x\mathrm{d}\tau\mathrm{d}\tau'\, \bar{\psi}_R(x,\tau)\psi_L(x,\tau)\bar{\psi}_L(x,\tau')\psi_R(x,\tau') \\ &-\frac{2D_b a_c}{u}\int \mathrm{d}x\mathrm{d}\tau\, \bar{\psi}_R(x,\tau)\psi_L(x,\tau)\bar{\psi}_L(x,\tau)\psi_R(x,\tau). \end{aligned} \tag{3.97}$$

The second term is now local in time and renormalizes g_1. However, for spinless fermions g_1 and g_2 processes are identical and we can write

$$\tilde{g}_2 = g_2 + 2D_b a_c/u. \tag{3.98}$$

In terms of the modified coupling constants the RG equations are

$$d\tilde{K}/dl = -\tilde{K}^2 D/2, \tag{3.99}$$

$$d\tilde{u}/dl = -\tilde{u}\tilde{K} D/2, \tag{3.100}$$

$$dD/dl = (3 - 2\tilde{K})D. \tag{3.101}$$

Here $\tilde{K}$ and $\tilde{u}$ are related to the modified interaction constant

$$\tilde{g}_2(l) = g_2(l) + \pi u D(l) \tag{3.102}$$

according to equation (3.69). The modification of the plasmon velocity u as well as its renormalization can be neglected in (3.102) since it would lead to higher orders terms in D. Therefore, the above RG equation for u decouples from the other two and we will neglect it in the following.

An important point is that $\tilde{K}$ does not describe interactions alone but actually contains admixture of disorder through the renormalized $\tilde{g}_2$. By expanding $\tilde{K}$ in D one obtains

$$\frac{dK}{dl} = -\frac{1}{2}\left[K^2 - \frac{(1+K^2)(3-2K)}{2}\right]D, \tag{3.103}$$

$$\frac{dD}{dl} = (3 - 2K)D, \tag{3.104}$$

where K is now the true LL constant. The result we obtain is that disorder renormalizes interactions and vice versa.

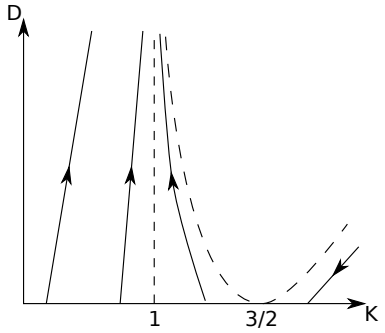

Figure 3.7.: Flow diagram of the Giamarchi-Schulz RG. D denotes the dimensionless disorder strength and K is the Luttinger parameter describing the strength of interactions.

The flow equations show us that interactions become less relevant in the localized regime because of the vanishing overlap between electron wavefunctions ("disorder kills interactions"). However, for $K > 3/2$ the effect of interactions reduces disorder and we are in a delocalized regime. Physically this can be explained through inelastic electron-electron scattering events that destroy the phase coherence and therefore suppress interference effects such as localization. The corresponding flow diagram is depicted in Fig. 3.7.

4. Nonequilibrium field theory

In this chapter we introduce the Keldysh formalism and derive the kinetic equation for helical fermions to the lowest order in the self energy. While we are only interested in transport properties close to equilibrium the derivation from a more fundamental theory will give us some insight into the assumptions we make. Furthermore, it is straightforward to continue the calculations by taking into account higher orders in the self energy. Concerning the literature we will mostly follow Ref. [17]. Another nice introduction can also be found in Ref. [39].

4.1. The closed time contour

The most important conceptual idea behind the Keldysh formalism is the Keldysh contour in the imaginary time plane. Given its importance we dedicate the whole first section to its discussion. Consider a quantum many body system governed by the time dependant Hamiltonian $\hat{H}(t)$.[1] Furthermore, let $\hat{\rho}(-\infty)$ be the density matrix that specifies the systems state in the distant past ($t = -\infty$) which we assume to be known (e.g. in thermal equilibrium). We assume that the original Hamiltonian $\hat{H}(-\infty)$ is non-interacting and interactions are switched on adiabatically. Additionally, the Hamiltonian may contain "true"[2] time dependance through external fields or boundary conditions. Therefore, $\hat{\rho}$ is driven away from equilibrium and evolves according to the Von Neumann equation,

$$\partial_t \hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)], \tag{4.1}$$

which is formally solved by introducing the unitary time evolution operator $\hat{\mathcal{U}}_{t,t'}$ as

$$\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)[\hat{\mathcal{U}}_{t,-\infty}]^\dagger = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\hat{\mathcal{U}}_{-\infty,t}. \tag{4.2}$$

Here,

$$\hat{\mathcal{U}}_{t,t'} = \mathbb{T}\exp\{-i\int_{t'}^{t} \mathrm{d}t_1 \hat{H}(t_1)\} \tag{4.3}$$

and $\mathbb{T}$ denotes time ordering.

[1] In this chapter we work in the Schrödinger picture, where operators are time independant while states and the density matrix depend on time.

[2] Since interactions are switched on adiabatically the potential is smooth as a function of time and therefore approximatelly constant $\hat{V}(t) \approx \hat{V}(t_0)$.

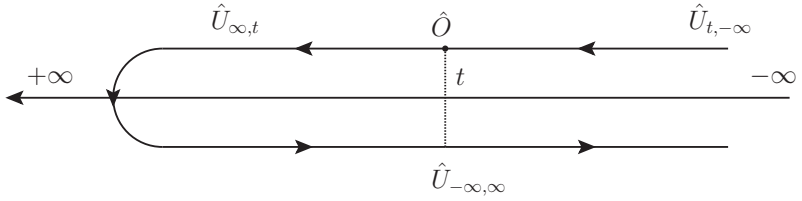

Figure 4.1.: The Keldysh contour in the imaginary time plane.

In order to make physical predictions we have to calculate the expectation value of operators:

$$\langle \hat{O} \rangle (t) = \frac{\mathrm{Tr}\{\hat{\rho}(t)\hat{O}\}}{\mathrm{Tr}\{\hat{\rho}(t)\}} = \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}} \mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\}. \tag{4.4}$$

Read from left to right this means that we have the evolution from $t = -\infty$ (where $\hat{\rho}$ is known) to time t where we measure $\hat{O}$ and then back to $t = -\infty$. To get a better understanding of how one can implement this evolution we first consider the equilibrium situation at zero temperature. Let $|\mathrm{GS}\rangle$ be the ground state of the interacting many body problem and $|0\rangle$ the ground state of the corresponding non interacting system. They are connected according to

$$|\mathrm{GS}\rangle = \hat{\mathcal{U}}_{t,-\infty}|0\rangle. \tag{4.5}$$

Since we are in equilibrium the only allowed time dependance of $\hat{H}$ is the adiabatic switching on of interaction such that there is no level crossing i.e. a system in the ground state $|0\rangle$ remains in the state $|0\rangle$ during the whole evolution and can pick up a phase factor χ at most:

$$\hat{\mathcal{U}}_{\infty,-\infty}|0\rangle = e^{i\chi}|0\rangle, \qquad e^{i\chi} = \langle 0|\hat{\mathcal{U}}_{\infty,-\infty}|0\rangle. \tag{4.6}$$

Therefore, we get

$$\begin{aligned}
\langle \mathrm{GS}|\hat{O}|\mathrm{GS}\rangle =& \langle 0|\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle && (4.7)\\
=& e^{-i\chi}\langle 0|e^{i\chi}\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle && (4.8)\\
=& e^{-i\chi}\langle 0|\hat{\mathcal{U}}_{\infty,-\infty}\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle && (4.9)\\
=& \frac{\langle 0|\hat{\mathcal{U}}_{\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle}{\langle 0|\hat{\mathcal{U}}_{\infty,-\infty}|0\rangle}. && (4.10)
\end{aligned}$$

Thus at the cost of an additional denominator we only have to consider forward evolution in time. However, the whole argument breaks down if the Hamiltonian contains for instance external fields with non adiabatic time dependendance. In general we have to consider both backward and forward evolution in time. Lets rewrite (4.4) by inserting $\hat{\mathbb{1}} = \hat{\mathcal{U}}_{t,\infty}\hat{\mathcal{U}}_{\infty,t}$ and using the fact that $\mathrm{Tr}\{\hat{\rho}(t)\} = \mathrm{Tr}\{\hat{\rho}(-\infty)\}$:

$$\langle \hat{O} \rangle (t) = \frac{1}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}} \mathrm{Tr}\{\hat{U}_{-\infty,\infty}\hat{\mathcal{U}}_{\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\}. \tag{4.11}$$

Eq. (4.11) describes time evolution along the Keldysh contour $\mathcal{C}$ in the imaginary time plane shown in Fig. 4.1. Let $\hat{\mathcal{U}}_{\mathcal{C}} = \hat{\mathcal{U}}_{-\infty,\infty}\hat{\mathcal{U}}_{\infty,-\infty}$ be the evolution operator along the Keldysh contour. Note that if the Hamiltonian is identical on both branches, time evolution along $\mathcal{C}$ brings *any* state back exactly to the original state (not even a phase is accumulated). Therefore, $\hat{\mathcal{U}}_{\mathcal{C}} = \hat{\mathbb{1}}$ and the partition function $Z \equiv \mathrm{Tr}\{\hat{\rho}(-\infty)\hat{\mathcal{U}}_{\mathcal{C}}\}/\mathrm{Tr}\{\hat{\rho}(-\infty)\}$ is identically

equal to unity, $Z = 1$.[3] In order to insert an observable along the path we have to add a source term

$$\hat{H}_V^{\pm}(t) = \hat{H}(t) \pm \hat{O}V(t) \tag{4.12}$$

where +(-) denote the forward (backward) branch of the contour. Then the Hamiltonian is different along the two branches and $\hat{\mathcal{U}}_{\mathcal{C}}[V] \neq \hat{\mathbb{1}}$. Observables are then calculated by varying the *generating function* Z[V],

$$\mathrm{Z[V]} = \frac{\mathrm{Tr}\{\hat{\rho}(-\infty)\hat{\mathcal{U}}_{\mathcal{C}}[\mathrm{V}]\}}{\mathrm{Tr}\,\{\hat{\rho}(-\infty)\}}, \tag{4.13}$$

as

$$\langle \hat{O} \rangle (t) = \frac{i}{2} \frac{\delta}{\delta V(t)} \,\mathrm{Z[V]}|_{V(t)=0}\,. \tag{4.14}$$

4.2. Implementation in the path integral formalism

In order to use the above discussed ideas in the real time path integral formalism, we have to swap the integration along the real time axis with the integration along the Keldysh contour $\int_{-\infty}^{\infty} \mathrm{d}t \to \int_{\mathcal{C}} \mathrm{d}t$ and double the number of our time dependant fields since fields on the both branches are distinct. The resulting field theory therefore gains a matrix structure in Keldysh space and we have to define additional propagators:

$$\langle \phi^+(t)\bar{\phi}^-(t')\rangle = iG^<(t,t') \qquad \text{lesser Green's function,} \tag{4.15}$$

$$\langle \phi^-(t)\bar{\phi}^+(t')\rangle = iG^>(t,t') \qquad \text{greater Green's function,} \tag{4.16}$$

$$\langle \phi^+(t)\bar{\phi}^+(t')\rangle = iG^{\mathbb{T}}(t,t') \qquad \text{time ordered Green's function,} \tag{4.17}$$

$$\langle \phi^-(t)\bar{\phi}^-(t')\rangle = iG^{\tilde{\mathbb{T}}}(t,t') \qquad \text{anti time ordered Green's function.} \tag{4.18}$$

Here ϕ is an arbitrary field (either bosonic or fermionic) and time ordering $\mathbb{T}$ puts the "latest (time) on the left" while anti time ordering $\tilde{\mathbb{T}}$ orders in the reverse order. Furthermore, +(-) denote the upper (lower) branch on the Keldysh contour.
It turns out that these Green's functions are not linearly independant, since

$$G^{\mathbb{T}}(t,t') + G^{\tilde{\mathbb{T}}}(t,t') - G^>(t,t') - G^<(t,t') = 0. \tag{4.19}$$

Therefore, one defines the Keldysh rotation for

(i) bosons:

$$\begin{pmatrix} \phi^{cl}(t) \\ \phi^{q}(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \phi^+(t) \\ \phi^-(t) \end{pmatrix}, \tag{4.20}$$

where cl(q) denotes the classical (quantum) component and the complex conjugate fields $\bar{\phi}$ transform in the same way.

(ii) fermions:

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi^+(t) \\ \psi^-(t) \end{pmatrix}, \tag{4.21}$$

$$\begin{pmatrix} \bar{\psi}_1(t) \\ \bar{\psi}_2(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{\psi}^+(t) \\ \bar{\psi}^-(t) \end{pmatrix}. \tag{4.22}$$

Since ψ,$\bar{\psi}$ are not complex conjugates but rather independant Grassman variables they can be transformed independantly from each other. The transformation is chosen such that the fermionic self energy has the same structure as the Green's function.

[3] This normalisation makes the Keldysh technique so useful for disordered systems.

With this rotation the Green's function in Keldysh space has one vanishing entry.

(i) Complex bosonic Green's function (hereafter $\alpha, \beta \in \{cl, q\}$):

$$D_{\alpha,\beta}(t,t') = -i\left\langle \varphi_\alpha(t)\bar{\varphi}_\beta(t')\right\rangle = \begin{pmatrix} D^K(t,t') & D^R(t,t') \\ D^A(t,t') & 0 \end{pmatrix}. \tag{4.23}$$

(ii) Fermionic Green's function ($a, b \in \{1, 2\}$):

$$G_{ab}(t,t') = -i\left\langle \psi_a(t)\bar{\psi}_b(t')\right\rangle = \begin{pmatrix} G^R(t,t') & G^K(t,t') \\ 0 & G^A(t,t') \end{pmatrix}. \tag{4.24}$$

R(A) denote the retarded (advanced) component while K stands for the Keldysh component. We conclude the discussion by summarising some properties of these propagators:

- $\left[G^R\right]^\dagger = G^A$ and $\left[G^K\right]^\dagger = -G^K$.
- While the retarded and advanced components contain information about the spectrum of the system, the Keldysh component carries information about its distribution function. In general we may parametrize the antihermitian matrix G^K with the help of a Hermitian matrix $F = F^\dagger$:

$$G^K = G^R \circ F - F \circ G^A. \tag{4.25}$$

  Here $\circ$ denotes summation over all "labels" of the functions e.g.

$$\left(\hat{A} \circ \hat{B}\right)(\mathbf{x}_1, \mathbf{x}_2) = \int \mathrm{d}\mathbf{x} \sum_\beta A(\mathbf{x}_1, \mathbf{x})^{\alpha,\beta} B(\mathbf{x}, \mathbf{x}_2)^{\beta,\gamma}, \tag{4.26}$$

  where $\mathbf{x}$ is a set of continous indices e.g. (x,t), greek letters imply additional quantum numbers and the hat denotes matrix structure in Keldysh space. The Wigner transform (see Sec. 4.3.1) $F(t,\epsilon)$ of F is referred to as the distribution function. For example in the case of free fermions we have $F(\epsilon) = (1 - 2f^{(0)}(\epsilon))$, where $f^{(0)}$ is the Fermi distribution.
- Due to the causality structure of retarded and advanced components of the propagator ($G^R(t,t') \sim \Theta(t-t')$ while $G^A(t,t') \sim \Theta(t'-t)$) certain products of retarded and advanced components vanish. For example $G^{R(A)}(t,t')G^{R(A)}(t',t)$ is always zero.

4.3. The kinetic equation

The Keldysh formalism presents a way to microscopically motivate the semiclassical kinetic equation known from statistical mechanics [40]. To derive this kinetic equation we start with the Dyson equation in Keldysh space:

$$\hat{G} = \hat{G}_0 + \hat{G}_0 \circ \hat{\Sigma} \circ \hat{G} \tag{4.27}$$

$$\Leftrightarrow \quad \left(\hat{G}_0^{-1} - \hat{\Sigma}\right) \circ \hat{G} = \hat{\mathbb{1}} \tag{4.28}$$

$$\Leftrightarrow \quad \begin{pmatrix} \left[G_0^R\right]^{-1} - \Sigma^R & -\Sigma^K \\ 0 & \left[G_0^A\right]^{-1} - \Sigma^A \end{pmatrix} \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} = \hat{\mathbb{1}} \tag{4.29}$$

$$\Leftrightarrow \quad \begin{pmatrix} \left(\left[G_0^R\right]^{-1} - \Sigma^R\right) G^R & \left(\left[G_0^R\right]^{-1} - \Sigma^R\right) G^K - \Sigma^K G^A \\ \left(\left[G_0^A\right]^{-1} - \Sigma^A\right) G^A & 0 \end{pmatrix} = \hat{\mathbb{1}}. \tag{4.30}$$

We read off the component that determines the Keldysh Green's function and parametrize it as $G^K = G^R \circ F - F \circ G^A$. Thus we end up with

$$\left(\left[G_0^R\right]^{-1} - \Sigma^R\right)\left(G^R \circ F - F \circ G^A\right) = \Sigma^K \circ G^A. \tag{4.31}$$

We multiply this equation with $[G_0^A]^{-1} - \Sigma^A$ from the right and use the other components of the Dyson equation:$([G_0^{R(A)}]^{-1} - \Sigma^{R(A)})G^{R(A)} = 1$ to get

$$F \circ \left[G_0^A\right]^{-1} - \left[G_0^R\right]^{-1} \circ F = \Sigma^K - \left(\Sigma^R \circ F - F \circ \Sigma^A\right). \tag{4.32}$$

Since (ommiting $\pm i0$) $G_0^A = G_0^R$, the expression takes the form

$$-\left[\left(G_0^R\right)^{-1} \circ F\right] = \Sigma^K - \left(\Sigma^R \circ F - F \circ \Sigma^A\right), \tag{4.33}$$

where $[A \circ B] = A \circ B - B \circ A$ is the commutator in Keldysh space. The LHS of (4.33) is called the kinetic term while the RHS is the collision integral.

4.3.1. The Wigner transformation

Since it is usually difficult to solve the kinetic equation (4.33) in full generality we have to make some physically motivated assumptions. We make use of the scale separation between intrinsic microscopic scales and extrinsic macroscopic scales dictated by e.g. an external potential. Mathematically this is most elegantly achieved by the Wigner transform (WT). Let us first introduce space time and energy momentum coordinates $\mathbf{x} = (x, t)$, $\mathbf{p} = (p, \epsilon)$ and the corresponding scalar products

$$\mathbf{xp} = xp - \epsilon t, \qquad \partial_{\mathbf{x}}\partial_{\mathbf{p}} = \partial_x\partial_p - \partial_t\partial_\epsilon. \tag{4.34}$$

Next consider a two point function $A(\mathbf{x}_1, \mathbf{x}_2)$. We make a transformation to relative and central coordinates,

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & -1\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}. \tag{4.35}$$

The Wigner transform is defined as the Fourier transform of the relative coordinates and we obtain a function of central coordinates $\mathbf{X}$ and relative momentum $\mathbf{p}$

$$A(\mathbf{X}, \mathbf{p}) = \int \mathrm{d}\mathbf{x}\, e^{-i\mathbf{px}} A(\mathbf{X} + \frac{\mathbf{x}}{2}, \mathbf{X} - \frac{\mathbf{x}}{2}). \tag{4.36}$$

If we assume that the central coordinates are macroscopic while the relative coordinates are the quantum coordinates, we can make the assumption $\delta\mathbf{X}\delta\mathbf{p} \gg 1$ where $\delta\mathbf{X}$, $\delta\mathbf{p}$ are characteristic scales on which $\mathbf{X}$, $\mathbf{p}$ vary.[4] This implies that the function A is a smooth function of the central coordinate $\mathbf{X}$ and fast varying in the relative momentum $\mathbf{p}$. Under these prerequisites it is possible to make some further simplifications. Consider for instance the convolution $C = A \circ B \Leftrightarrow C(\mathbf{x}_1, \mathbf{x}_2) = \int \mathrm{d}\mathbf{x}_3\, A(\mathbf{x}_1, \mathbf{x}_3)B(\mathbf{x}_3, \mathbf{x}_2)$ which transforms to

$$C(\mathbf{X}, \mathbf{p}) = A(\mathbf{X}, \mathbf{p}) e^{\frac{i}{2}\left(\overleftarrow{\partial_{\mathbf{X}}}\overrightarrow{\partial_{\mathbf{p}}} - \overleftarrow{\partial_{\mathbf{p}}}\overrightarrow{\partial_{\mathbf{X}}}\right)} B(\mathbf{X}, \mathbf{p}). \tag{4.37}$$

[4] One could ask if there is a similar requirement for the time domain i.e. $\delta T\delta\epsilon \gg 1$. This would be rather restricting since in the low temperature regime energy fluctuations are typically of the order of temperature $\delta\epsilon \sim \beta^{-1}$ (because of notational issues T is the central time coordinate and β is inverse temperature). However, this would only allow us to treat processes with $\delta T \gg \beta$. Fortunately, the energy is normally locked to the quasiparticle energy $\tilde{\xi}_k$ and has no dynamics of its own as long as the quasiparticle peak is sharp. The actual criterion is therefore $\delta\epsilon \gg \tau_{qp}^{-1}$ or $\tau_{qp} \gg \beta$ which is fullfilled in most systems. As we will see, we also have to demand this relation for applying the quasiparticle approximation below.

Under the assumption $\delta\mathbf{X}\delta\mathbf{p} \gg 1$ the operator $\partial_X\partial_p$ is "small" and we may expand the exponential to gain

$$C(\mathbf{X},\mathbf{p}) \approx A(\mathbf{X},\mathbf{p})B(\mathbf{X},\mathbf{p}) + \frac{i}{2}\left(\partial_{\mathbf{X}}A\partial_{\mathbf{p}}B - \partial_{\mathbf{p}}A\partial_{\mathbf{X}}B\right). \tag{4.38}$$

From this we immediatedly get a approximation for the commutator

$$[A \overset{\circ}{,} B] \to i\left(\partial_{\mathbf{X}}A\partial_{\mathbf{p}}B - \partial_{\mathbf{p}}A\partial_{\mathbf{X}}B\right). \tag{4.39}$$

4.4. Derivation of the kinetic equation for the HLL

Let us now apply the developed theory to derive the kinetic equation for the helical Luttinger liquid. To lighten the notation, summation convention over recurring indices is implied.

4.4.1. The action

The real time action describing our model from Sec. 2.2.4 is

$$S_0[\psi,\bar{\psi}] = \sum_{\eta=R,L}\int_{\mathcal{C}}\mathrm{d}t\int\mathrm{d}x\,\bar{\psi}_\eta\left(i\partial_t + i\eta\partial_x - E^{cl}(x,t)\right)\psi_\eta, \tag{4.40}$$

$$S_{int}[\psi,\bar{\psi}] = -\sum_{\sigma,\sigma'}\int_{\mathcal{C}}\mathrm{d}t\int\mathrm{d}x\,\mathrm{d}x'\,\rho_\sigma(x,t)V(x-x')\rho_{\sigma'}(x',t), \tag{4.41}$$

$$S_I[\psi,\bar{\psi}] = -\sum_{\sigma}\int_{\mathcal{C}}\mathrm{d}t\int\mathrm{d}x\,U(x)\rho_\sigma(x,t). \tag{4.42}$$

Here we added a classical source field $E^{cl}(x,t)$ that plays the role of external electric field and the density is defined as

$$\rho_\sigma(x,t) = \bar{\psi}_\sigma(x,t)\psi_\sigma(x,t). \tag{4.43}$$

We introduce the momentum dependant transformation to the chiral basis,

$$\psi_{\sigma,k} = B_k^{\sigma,\eta}\psi_{\eta,k}, \tag{4.44}$$

$$\bar{\psi}_{\sigma,k} = B_k^{\eta,\sigma}\bar{\psi}_{\eta,k}, \tag{4.45}$$

with the matrix B_k defined in (2.33). Furthermore, we use the following convention for Fourier transform

$$\psi_\sigma(x,t) = \frac{1}{L}\sum_k\int\frac{\mathrm{d}\epsilon}{2\pi}\,\psi_\sigma(k,\epsilon)e^{i(kx-\epsilon t)}. \tag{4.46}$$

If we make the transformation to the chiral basis and momentum space the resulting terms are

$$S_{int}[\psi,\bar{\psi}] = -\frac{1}{L}\sum_q\int_{\mathcal{C}}\mathrm{d}t\,V_q\left|\rho_q(t)\right|^2, \tag{4.47}$$

$$S_I[\psi,\bar{\psi}] = -\frac{1}{L}\sum_q\int_{\mathcal{C}}\mathrm{d}t\,U_q\rho_q(t). \tag{4.48}$$

Here we defined

$$\rho_q \equiv \frac{1}{L}\sum_k\sum_{\eta_1\eta_2}\bar{\psi}_{\eta_1,k}\psi_{\eta_2,k-q}\left[B_kB_{k-q}\right]^{\eta_1\eta_2} \tag{4.49}$$

and used the fact that $\rho_q^* = \rho_{-q}$ since $\rho(x)$ is real.

We proceed by decoupling the interaction term via a complex Hubbard Stratonovich transformation,

$$e^{-i\int_{\mathcal{C}}\mathrm{d}t\,\frac{V_0}{L}\sum_q|\rho_q|^2} = \int \mathrm{D}[\varphi]\, e^{-i\int_{\mathcal{C}}\mathrm{d}t\,\frac{1}{L}\sum_q\left[V_0^{-1}|\varphi_q(t)|^2-\rho_q^*(t)\varphi_q(t)-\rho_q(t)\bar{\varphi}_q(t)\right]}. \tag{4.50}$$

Keldysh rotation

Let us now employ the rotation to Keldysh space Eqs. (4.20),(4.21) and (4.22). The source field E transforms as the bosonic field φ. We demonstrate how the transformation works for the example of the free boson action (arguments of the fields are dropped for simplicity)

$$\begin{aligned}
\int_{\mathcal{C}}\mathrm{d}t\,\bar{\varphi}\varphi &= \int_{-\infty}^{\infty}\mathrm{d}t\,\left(\bar{\varphi}^+\varphi^+ - \bar{\varphi}^-\varphi^-\right) && (4.51)\\
&\to \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\,\left(\left[\bar{\varphi}^{cl}+\bar{\varphi}^q\right]\left[\varphi^{cl}+\varphi^q\right] - \left[\bar{\varphi}^{cl}-\bar{\varphi}^q\right]\left[\varphi^{cl}-\varphi^q\right]\right) && (4.52)\\
&= \int_{-\infty}^{\infty}\mathrm{d}t\,\left(\bar{\varphi}^{cl}\varphi^q + \bar{\varphi}^q\varphi^{cl}\right) && (4.53)\\
&= \int_{-\infty}^{\infty}\mathrm{d}t\,\bar{\varphi}^\alpha\sigma^1_{\alpha,\beta}\varphi^\beta. && (4.54)
\end{aligned}$$

Proceeding analogously for the other terms we arrive at the action in Keldysh space:

$$\begin{aligned}
S_0 &= \sum_\eta\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{1}{L}\sum_k\bar{\psi}_{k,\eta,a}\left[G_0^{-1}\right]^{ab}_{k,\eta}\psi_{k,\eta,b}, && (4.55)\\
S_{int} &= \int_{-\infty}^{\infty}\mathrm{d}t\,\frac{1}{L}\sum_q\left[V_q^{-1}\bar{\varphi}_{\alpha,q}(t)\sigma^{1,\alpha,\beta}\varphi_{\beta,q}(t) - \rho^{ab}_{-q}\hat{\gamma}^\alpha_{ab}\varphi_{\alpha,q}(t) - \rho^{ab}_q\hat{\gamma}^\alpha_{ab}\bar{\varphi}_{\alpha,q}(t)\right], && (4.56)\\
S_I &= -\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{1}{L}\sum_q U_q\rho^{ab}_q\hat{\gamma}^{cl}_{ab}. && (4.57)
\end{aligned}$$

Here we defined

$$\begin{aligned}
\left[G_0^{-1}\right]^{ab}_{k,\eta} &= \begin{pmatrix} i\partial_t - \eta k - E_k^{cl}(t) + i0 & 0 \\ 0 & i\partial_t - \eta k - E_k^{cl}(t) - i0\end{pmatrix}, && (4.58)\\
\rho^{ab}_q(t) &= \frac{1}{L}\sum_k\sum_{\eta,\eta'}\bar{\psi}^a_{\eta_1,k}(t)\psi^b_{\eta_2,k-q}(t)\left[B_kB_{k-q}\right]^{\eta\eta'}, && (4.59)\\
\hat{\gamma}^{cl} &= \begin{pmatrix}1&0\\0&1\end{pmatrix},\ \hat{\gamma}^q = \begin{pmatrix}0&1\\1&0\end{pmatrix}. && (4.60)
\end{aligned}$$

Note that the free propagator is not really diagonal in Keldysh space because one normally has an infinitesimal $[G_0^{-1}]^K$, which we neglected here since a finite contribution will arise from the Keldysh component of the self energy. However, the propagator is diagonal in chiral space which considerably simplifies the treatment in the following.

Energy momentum representation

If we Fourier transform to energy space we have

$$\begin{aligned}
S_0 &= \sum_\eta\sum_{\mathbf{k}}\bar{\psi}_{\mathbf{k},\eta,a}\left[G_0^{-1}\right]^{ab}_{\eta,\mathbf{k}}\psi_{\mathbf{k},\eta,b}, && (4.61)\\
S_{int} &= \sum_{\mathbf{q}}\left[V_q^{-1}\bar{\varphi}_{\alpha,\mathbf{q}}\sigma^{1,\alpha,\beta}\varphi_{\beta,\mathbf{q}} - \rho^{ab}_{-\mathbf{q}}\hat{\gamma}^\alpha_{ab}\varphi_{\alpha,\mathbf{q}} - \rho^{ab}_{\mathbf{q}}\hat{\gamma}^\alpha_{ab}\bar{\varphi}_{\alpha,\mathbf{q}}\right], && (4.62)\\
S_I &= -\sum_{\mathbf{q}}U_q\rho^{ab}_{\mathbf{q}}\hat{\gamma}^{cl}_{ab} && (4.63)
\end{aligned}$$

Where we defined the following objects:

$$\left[G_0^{-1}\right]^{ab}_{\eta,\mathbf{k}} = \begin{pmatrix} \epsilon - \eta k - E^{cl}_{\mathbf{k}} + i0 & 0 \\ 0 & \epsilon - \eta k - E^{cl}_{\mathbf{k}} - i0 \end{pmatrix}, \tag{4.64}$$

$$\rho^{ab}_{\mathbf{q}} = \sum_{\eta_1,\eta_2} \sum_{\mathbf{k}} \bar{\psi}^a_{\eta_1,\mathbf{k}} \psi^b_{\eta_2,\mathbf{k}-\mathbf{q}} \left[B_k B_{k-q}\right]^{\eta_1\eta_2}. \tag{4.65}$$

For convenience we also introduced the 2 momentum

$$\mathbf{k} = (k,\epsilon), \qquad \sum_{\mathbf{k}} = \frac{1}{L}\sum_k \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi}. \tag{4.66}$$

Finally, the free bosonic propagator is given by

$$iD^{\alpha\beta}_{0,\mathbf{q}} = \left\langle \varphi^\alpha_{\mathbf{q}} \bar{\varphi}^\beta_{\mathbf{q}} \right\rangle_0 = \begin{pmatrix} 0 & V_q^{-1} \\ V_q^{-1} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & V_q \\ V_q & 0 \end{pmatrix} \equiv \begin{pmatrix} D_0^K & D_0^R \\ D_0^A & 0 \end{pmatrix}. \tag{4.67}$$

4.4.2. Wigner transformation of the kinetic equation

The left-hand side of Eq. 4.33 reads

$$-\left[\left(G_0^R\right)^{-1} \circ F\right] = -\left[i\partial_t + i\eta\partial_x - E^{cl}(X,T) \circ F\right]. \tag{4.68}$$

For translation invariant operators like $i\partial_t, i\partial_x$ the Wigner transform is just ϵ, p while the external potential depends only on the macroscopic scales and is thus its own WT. Therefore, we have (using the properties of WT and $\partial_{\mathbf{X}}\partial_{\mathbf{p}} = \partial_X\partial_p - \partial_T\partial_\epsilon$):

$$[i\partial_t; F] = i\partial_\epsilon\epsilon\partial_T F = i\partial_T F, \tag{4.69}$$

$$[i\partial_x; F] = i\partial_p p\partial_X F = i\partial_X F, \tag{4.70}$$

$$\left[E^{cl}(X,T); F\right] = i\partial_{\mathbf{X}} E^{cl}(\mathbf{X})\partial_{\mathbf{p}} F = i\partial_X E^{cl}(\mathbf{X})\partial_p F - i\partial_T E^{cl}(\mathbf{X})\partial_\epsilon F. \tag{4.71}$$

Thus, the term (4.68) becomes

$$-i\left(\partial_T + \partial_X - \partial_X E^{cl}(\mathbf{X})\partial_p + \partial_T E^{cl}(\mathbf{X})\partial_\epsilon\right) F \tag{4.72}$$

For the right-hand side of Eq. (4.33) the properties of WT give

$$\begin{aligned} &\Sigma^K - \left(\Sigma^R \circ F - F \circ \Sigma^A\right), \\ \to\ &\Sigma^K - F\left(\Sigma^R - \Sigma^A\right) - i\partial_{\mathbf{X}}\,\mathfrak{Re}(\Sigma^R)\partial_{\mathbf{p}} F + i\partial_{\mathbf{p}}\,\mathfrak{Re}(\Sigma^R)\partial_{\mathbf{X}} F, \end{aligned} \tag{4.73}$$

where we used the fact that $\left(\Sigma^A\right)^* = \Sigma^R$. Combining both parts yields

$$\begin{aligned} &\Big((1 - \partial_\epsilon\,\mathfrak{Re}\,\Sigma^R)\partial_T + (1 - \partial_p\,\mathfrak{Re}\,\Sigma^R)\partial_X - \partial_X\tilde{E}^{cl}(\mathbf{X})\partial_p + \partial_T\tilde{E}^{cl}(\mathbf{X})\partial_\epsilon\Big) F \\ &= i\Sigma^K - iF\left(\Sigma^R - \Sigma^A\right), \end{aligned} \tag{4.74}$$

where $\tilde{E}^{cl} = E^{cl} + \mathfrak{Re}\,\Sigma^R$. In a static ($\partial_T = 0$) situation any distribution function $F(\epsilon)$ that depends on energy only, nullifies the LHS. One can show that there is also a special equilibrium solution that nullifies the collision kernel e.g. for fermions $F^{eq} = \tanh(\epsilon - \mu)/2T$. To make progress away from equilibrium we shift the energy argument of the distribution function:

$$F(R,T,p,\epsilon) = \tilde{F}(R,T,p,\epsilon - \xi_p - \tilde{E}). \tag{4.75}$$

One can show that this function obeys a similar kinetic equation as (4.74) only that there is no energy derivative. Now we employ the so called quasiparticle approximation. $\tilde{F}$ always appears in combination with the spectral weight which is proportional to $G^R - G^A$ and is a strongly peaked function around $\tilde{\epsilon} = \epsilon - \xi_p - \tilde{E}$ with width of the inverse quasiparticle lifetime τ_{qp}^{-1}. If the characteristic energy scale $\delta\tilde{\epsilon}$ on which the distribution function $F(\tilde{\epsilon})$ variies is much larger than the inverse particle lifetime $\delta\tilde{\epsilon} \gg \tau_{qp}^{-1}$ one can disregard the energy dependance of the distribution function

$$\tilde{F}(X,T,p,\tilde{\epsilon}) \approx \tilde{F}(X,T,p,0) \equiv \tilde{F}(X,T,p). \tag{4.76}$$

The resulting mass shell distribution function $\tilde{F}(R,T,p)$ is essentially a classical object and can be regarded as the time dependant probability to find a particle at a given point in phase space (X,p). All observables are determined by $\tilde{F}$ to leading order in the small parameter $(\delta\tilde{\epsilon}\tau_{qp})^{-1}$. Therefore, the kinetic equation describes a semiclassical approximation of the full quantum description. Quantum Mechanics modifies the dispersion relation, the effective potential and the quasiparticle weight as well as the collisions integral. For now we neglect the renormalization through the self energy on the LHS and assume the external potential as $E^{cl} = eE(T)X$ to get

$$\left(\partial_T + \partial_X - eE(T)\partial_p\right) F = i\Sigma^K - iF\left(\Sigma^R - \Sigma^A\right). \tag{4.77}$$

Thus the LHS is exactly the same as for a phenomenological kinetic equation. For the RHS, we need to calculate the corresponding self energies, which will be done in the next section.

4.5. Keldysh objects

4.5.1. Self energy due to interactions

We obtain the self energy from the Dyson equation, Eq. (4.27). The full propagator is given by

$$G^{ab}_{\eta,\mathbf{k}} = -i\int \mathrm{D}[\psi,\varphi]\,\psi^a_{\eta,\mathbf{k}}\bar{\psi}^b_{\eta,\mathbf{k}} e^{iS_0} e^{iS_{int}} \tag{4.78}$$

$$= -i\left\langle \left(1 + iS_{int} + \frac{i^2}{2}S^2_{int} + \cdots\right)\psi^a_{\eta,\mathbf{k}}\bar{\psi}^b_{\eta,\mathbf{k}}\right\rangle_0 \tag{4.79}$$

Since the interaction is linear in the bosonic field φ the first nonvanishing order is the second one,

$$\frac{i}{2}\left\langle \left(\rho^{cd}_{\mathbf{p}}\gamma^\alpha_{cd}\bar{\varphi}^\alpha_{\mathbf{p}} + \rho^{cd}_{-\mathbf{p}}\gamma^\alpha_{cd}\varphi^\alpha_{\mathbf{p}}\right)\left(\rho^{ef}_{\mathbf{p}'}\gamma^\beta_{ef}\bar{\varphi}^\beta_{\mathbf{p}'} + \rho^{ef}_{-\mathbf{p}'}\gamma^\beta_{ef}\varphi^\beta_{\mathbf{p}'}\right)\psi^a_{\eta,\mathbf{k}}\bar{\psi}^b_{\eta,\mathbf{k}}\right\rangle_0 \tag{4.80}$$

$$= \frac{i}{2}\left\langle \rho^{cd}_{-\mathbf{p}}\rho^{ef}_{\mathbf{p}'}\psi^a_{\eta,\mathbf{k}}\bar{\psi}^b_{\eta,\mathbf{k}}\right\rangle_0 \gamma^\alpha_{cd}\gamma^\beta_{ef}\left\langle \bar{\varphi}^\alpha_{\mathbf{p}}\varphi^\beta_{\mathbf{p}'}\right\rangle_0 + \frac{i}{2}\left\langle \rho^{cd}_{\mathbf{p}}\rho^{ef}_{-\mathbf{p}'}\psi^a_{\eta,\mathbf{k}}\bar{\psi}^b_{\eta,\mathbf{k}}\right\rangle_0 \gamma^\alpha_{cd}\gamma^\beta_{ef}\left\langle \varphi^\alpha_{\mathbf{p}}\bar{\varphi}^\beta_{\mathbf{p}'}\right\rangle_0 \tag{4.81}$$

$$= i\left\langle \rho^{cd}_{-\mathbf{p}}\rho^{ef}_{\mathbf{p}}\psi^a_{\eta,\mathbf{k}}\bar{\psi}^b_{\eta,\mathbf{k}}\right\rangle_0 \gamma^\alpha_{cd}\gamma^\beta_{ef}\;\; iD^{\beta,\alpha}_{0,p}. \tag{4.82}$$

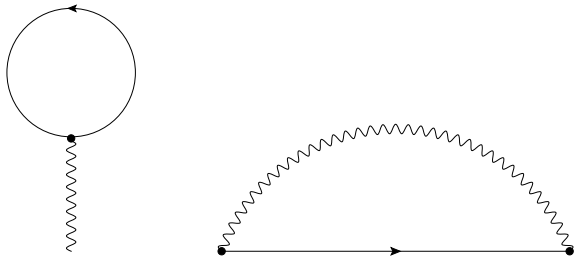

Figure 4.2.: Hartree (left) and Fock (right) contribution to the self energy in the lowest order.

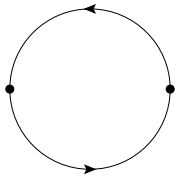

Figure 4.3.: Polarisation operator Π_0.

In the last line we made use of the fact that the free boson propagator is diagonal in momentum space and symmetric in Keldysh space, i.e. $D^{\alpha,\beta}_{0,p} = D^{\beta,\alpha}_{0,p}$.

In the lowest order only the Hartree and Fock diagrams (see Fig. 4.2) contribute. Since the Hartree diagrams cancel with the ionic background we concentrate only on the Fock contributions to the self energy which give

$$\Sigma^{ab(1)}_{\eta,\mathbf{k}} = 2i\tilde{B}^{\eta,\eta'}_{k,k+q}\left(\hat{\gamma}^{\alpha}\hat{G}_{\mathbf{k+q},\eta'}\hat{\gamma}^{\beta}\right)^{ab} D^{\beta,\alpha}_{0,q} \tag{4.83}$$

$$\begin{aligned}=2i\tilde{B}^{\eta,\eta'}_{k,k+q}\Big[&\left(\hat{\gamma}^{cl}\hat{G}_{\mathbf{k+q},\eta'}\hat{\gamma}^{cl}\right)^{ab} D^{K}_{0,q} + \left(\hat{\gamma}^{q}\hat{G}_{\mathbf{k+q},\eta'}\hat{\gamma}^{cl}\right)^{ab} D^{R}_{0,q} \\ &+ \left(\hat{\gamma}^{cl}\hat{G}_{\mathbf{k+q},\eta'}\hat{\gamma}^{q}\right)^{ab} D^{A}_{0,q}\Big].\end{aligned} \tag{4.84}$$

Here, we defined $\tilde{B}^{\eta,\eta'}_{k,k+q} \equiv [B^{\dagger}_{k}B_{k+q}]^{\eta\eta'}[B^{\dagger}_{k+q}B_{k}]^{\eta'\eta}$. Written in components this is

$$\Sigma^{R(A)}_{\eta,\mathbf{k}} = 2i\tilde{B}^{\eta,\eta'}_{k,k+q}\left[G^{R(A)}_{\mathbf{k+q},\eta'}D^{K}_{0,q} + G^{K}_{\mathbf{k+q},\eta'}D^{A(R)}_{0,q}\right], \tag{4.85}$$

$$\begin{aligned}\Sigma^{K}_{\eta,\mathbf{k}} =&2i\tilde{B}^{\eta,\eta'}_{k,k+q}\left[G^{K}_{\mathbf{k+q},\eta'}D^{K}_{0,q} + G^{R}_{\mathbf{k+q},\eta'}D^{A}_{0,q} + G^{A}_{\mathbf{k+q},\eta'}D^{R}_{0,q}\right] \\ =&2i\tilde{B}^{\eta,\eta'}_{k,k+q}\left[G^{K}_{\mathbf{k+q},\eta'}D^{K}_{0,q} - \left(G^{R}_{\mathbf{k+q},\eta'} - G^{A}_{\mathbf{k+q},\eta'}\right)\left(D^{R}_{0,q} - D^{A}_{0,q}\right)\right],\end{aligned} \tag{4.86}$$

where we made use of the fact that $G^{R(A)}_{\mathbf{k+q},\eta'}D^{R(A)}_{0,q} = 0$, due to the causality structure discussed in Sec. 4.2.

4.5.2. Polarisation operator

We obtain the polarization operator by calculating the correction to the boson propagator:

$$\left\langle \varphi^{\alpha}_{\mathbf{q}}\bar{\varphi}^{\beta}_{\mathbf{q}}\right\rangle = \left\langle \varphi^{\alpha}_{\mathbf{q}}\bar{\varphi}^{\beta}_{\mathbf{q}}e^{iS_{int}}\right\rangle_0 \tag{4.87}$$

$$\approx \left\langle \varphi^{\alpha}_{\mathbf{q}}\bar{\varphi}^{\beta}_{\mathbf{q}}\right\rangle_0 + \frac{i^2}{2}\left\langle \varphi^{\alpha}_{\mathbf{q}}\bar{\varphi}^{\beta}_{\mathbf{q}}S^2_{int}\right\rangle_0, \tag{4.88}$$

which yields

$$D^{\alpha,\beta}_{\mathbf{q}} = D^{\alpha,\beta}_{0,\mathbf{q}} - \frac{i^3}{2}\left\langle \varphi^{\alpha}_{\mathbf{q}}\bar{\varphi}^{\beta}_{\mathbf{q}}S^2_{int}\right\rangle_0. \tag{4.89}$$

The first order correction to the free bosonic propagator is the bare polarisation operator Π_0 shown in Fig. 4.3 and given by

$$\frac{i}{2}\left\langle \varphi^{\alpha}_{\mathbf{q}}\bar{\varphi}^{\beta}_{\mathbf{q}}\rho^{ab}_{-\mathbf{k}}\gamma^{\delta}_{ab}\varphi^{\delta}_{\mathbf{k}}\rho^{cd}_{\mathbf{p}}\gamma^{\epsilon}_{cd}\bar{\varphi}^{\epsilon}_{\mathbf{p}}\right\rangle_0 + \frac{i}{2}\left\langle \varphi^{\alpha}_{\mathbf{q}}\bar{\varphi}^{\beta}_{\mathbf{q}}\rho^{ab}_{\mathbf{k}}\gamma^{\delta}_{ab}\bar{\varphi}^{\delta}_{\mathbf{k}}\rho^{cd}_{-\mathbf{p}}\gamma^{\epsilon}_{cd}\varphi^{\epsilon}_{\mathbf{p}}\right\rangle_0. \tag{4.90}$$

Adding all contributions together the result is

$$\Pi^{\epsilon\delta}_{0}(\mathbf{q}) = i\tilde{B}^{\eta_1,\eta_2}_{p_1-q,p_1}\mathrm{Tr}_K\left\{\hat{G}_{\eta_1,\mathbf{p}_1-\mathbf{q}}\hat{\gamma}^{\delta}\hat{G}_{\eta_2,\mathbf{p}_1}\hat{\gamma}^{\epsilon}\right\}, \tag{4.91}$$

where Tr_K denotes the trace in Keldysh space. We can also write out the components explicitly:

$$\Pi_0^{A(R)}(\mathbf{q}) = i\left(G^{R(A)}_{\eta_1,\mathbf{p}_1-\mathbf{q}}G^{K}_{\eta_2,\mathbf{p}_1} + G^{K}_{\eta_1,\mathbf{p}_1-\mathbf{q}}G^{A(R)}_{\eta_2,\mathbf{p}_1}\right)\tilde{B}^{\eta_1,\eta_2}_{p_1-q,p_1}, \tag{4.92}$$

$$\Pi_0^{K}(\mathbf{q}) = i\left(G^{K}_{\eta_1,\mathbf{p}_1-\mathbf{q}}G^{K}_{\eta_2,\mathbf{p}_1} + G^{R}_{\eta_1,\mathbf{p}_1-\mathbf{q}}G^{A}_{\eta_2,\mathbf{p}_1} + G^{A}_{\eta_1,\mathbf{p}_1-\mathbf{q}}G^{R}_{\eta_2,\mathbf{p}_1}\right)\tilde{B}^{\eta_1,\eta_2}_{p_1-q,p_1}. \tag{4.93}$$

Now we use the free propagators $G^{(0)}$ defined as

$$G^{R(A),(0)}_{\eta,\mathbf{k}} = (\epsilon - \epsilon_{k,\eta} \pm i0)^{-1}, \tag{4.94}$$

$$G^{K,(0)}_{\eta,\mathbf{k}} = -2\pi F(\epsilon)\delta(\epsilon - \epsilon_{k,\eta}) \tag{4.95}$$

and integrate over energy using the arising delta functions. Furthermore, we may use that due to causality

$$G^R_{\mathbf{k}}G^A_{\mathbf{p}} + G^A_{\mathbf{k}}G^R_{\mathbf{p}} = -(G^R_{\mathbf{k}} - G^A_{\mathbf{k}})(G^R_{\mathbf{p}} - G^A_{\mathbf{p}}) = -[2\,\mathfrak{Im}\,G^R_{\mathbf{k}}]^2. \tag{4.96}$$

Note that in F we supressed the dependance on the central coordinates (R, T). The resulting expressions for the polarisation operator are

$$\Pi_0^{A(R)} = \frac{1}{L}\sum_p \tilde{B}^{\eta_1,\eta_2}_{p-q,p}\left(\frac{F(\epsilon_{p,\eta_2}) - F(\epsilon_{p-q,\eta_1})}{\epsilon_{p,\eta_2} - \omega - \epsilon_{p-q,\eta_1} \pm i0}\right), \tag{4.97}$$

$$\Pi_0^{K} = -\frac{2\pi i}{L}\sum_p \tilde{B}^{\eta_1,\eta_2}_{p-q,p}\delta(\epsilon_{p,\eta_2} - \omega - \epsilon_{p-q,\eta_1})\Big(F(\epsilon_{p,\eta_2})F(\epsilon_{p-q,\eta_1}) - 1\Big). \tag{4.98}$$

4.6. The Kinetic equation for interacting helical fermions

We now derive the kinetic equation for our specific case. First we use the self energies (4.85) and (4.86) and replace the propagators by

$$G^K_{\mathbf{k},\eta} = \left(G^R_{\mathbf{k},\eta} - G^A_{\mathbf{k},\eta}\right)F_{\mathbf{k},\eta}, \tag{4.99}$$

$$\left(G^R_{\mathbf{k},\eta} - G^A_{\mathbf{k},\eta}\right) = -2\pi i\delta(\epsilon - \epsilon_{k,\eta}). \tag{4.100}$$

Using these expressions we obtain the following expression for the terms in the collision integral calculated in Eq. (4.77):

$$\begin{aligned} i\Sigma^K_{\mathbf{k},\eta} =& \frac{2i}{L}\sum_{k_2}\tilde{B}^{\eta,\eta_2}_{k,k_2}\Big(F(\epsilon_{k_2,\eta_2})D^K_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta}) \\ &- \left[D^R_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta}) - D^A_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta})\right]\Big), \end{aligned} \tag{4.101}$$

and

$$\begin{aligned} iF_{\mathbf{k},\eta}\left(\Sigma^R_{\mathbf{k},\eta} - \Sigma^A_{\mathbf{k},\eta}\right) =& \frac{2i}{L}\sum_{k_2}\tilde{B}^{\eta,\eta_2}_{k,k_2}\Big(F(\epsilon_{k,\eta})D^K_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta}) + F(\epsilon_{k,\eta}) \\ &\times F(\epsilon_{k_2,\eta_2})\left[D^A_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta}) - D^R_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta})\right]\Big). \end{aligned} \tag{4.102}$$

Therefore, we arrive at the expression

$$\begin{aligned} I_{coll}[F_{\mathbf{k},\eta}] = \frac{2i}{L}\sum_{k_2}\tilde{B}^{\eta,\eta_2}_{k,k_2}\Big(&\left[F(\epsilon_{k_2,\eta_2}) - F(\epsilon_{k,\eta})\right]D^K_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta}) \\ &+ \left[F(\epsilon_{k_2,\eta_2})F(\epsilon_{k,\eta}) - 1\right]\left[D^R_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta}) - D^A_{0,k_2-k}(\epsilon_{k_2,\eta_2} - \epsilon_{k,\eta})\right]\Big). \end{aligned} \tag{4.103}$$

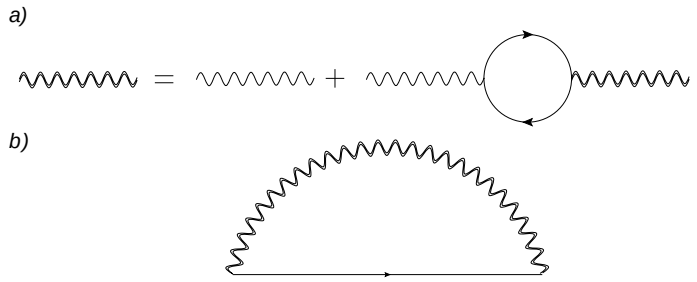

Figure 4.4.: a) self consistent equation describing the RPA propagator and b) the self energy diagram we take into account in deriving the kinetic equation.

If the interaction in Eq. (4.103) were the bare interaction, we would have $D^R_{0,k} = D^A_{0,k} = V_0$ and $D^K_{0,k} = 0$. Therefore, the collision integral would vanish and we would have no relaxation. Consequently, we have to include screening of interaction by the random phase approximation (RPA) shown in Fig. 4.4. In general the components of the RPA screened polarisation matrix can be written in the form (see Ref. [17]):

$$V^{R(A)}_{RPA} = |V^R_{RPA}|^2 \, (V_0^{-1} + \Pi_0^{A(R)}), \qquad V^K_{RPA} = -\,|V^R_{RPA}|^2 \, \Pi^K_0 . \tag{4.104}$$

With the explicit form of Π_0 from section 4.5.2 we get

$$(V^R_{RPA} - V^A_{RPA})_q(\omega) = -\,2i\,|V^R_{RPA}|^2 \, \mathfrak{Im}\,\Pi^R_0 \tag{4.105}$$

$$\begin{aligned} &= -\,\frac{2\pi i}{L}\,|V^R_{RPA}|^2_q\,(\omega) \sum_p \tilde{B}^{\eta_1,\eta_2}_{p-q,p}\,(F(\epsilon_{p,\eta_2}) - F(\epsilon_{p-q,\eta_1})) \\ &\quad \times \delta(\epsilon_{p,\eta_2} - \omega - \epsilon_{p-q,\eta_1}), \end{aligned} \tag{4.106}$$

$$\begin{aligned} -\,|V^R_{RPA}|^2 \, \Pi^K_0 =& \frac{2\pi i}{L}\,|V^R_{RPA}|^2_q\,(\omega) \sum_p \tilde{B}^{\eta_1,\eta_2}_{p-q,p}\,(F(\epsilon_{p,\eta_2})F(\epsilon_{p-q,\eta_1}) - 1) \\ &\times \delta(\epsilon_{p,\eta_2} - \omega - \epsilon_{p-q,\eta_1}). \end{aligned} \tag{4.107}$$

Exchanging the bare interaction D_0 in our previous collision integral (4.103) with the RPA screened interaction in Eq. (4.106) and (4.107) we get

$$\begin{aligned} I_{coll}[F_{\mathbf{k},\eta}] =& \frac{2i}{L}\frac{2\pi i}{L} \sum_{p,q} |V^R_{RPA,q}(\epsilon_{k+q,\eta_1} - \epsilon_{k,\eta})|^2 \, \tilde{B}^{\eta,\eta_2}_{k,k+q} \tilde{B}^{\eta_3,\eta_1}_{p-q,p} \\ &\times \delta(\epsilon_{k,\eta} + \epsilon_{p,\eta_1} - \epsilon_{k+q,\eta_2} - \epsilon_{p-q,\eta_3}) \Big([F(\epsilon_{k+q,\eta_2}) - F(\epsilon_{k,\eta})] \\ &\times [F(\epsilon_{p,\eta_1})F(\epsilon_{p-q,\eta_3}) - 1] - [F(\epsilon_{k,\eta})F(\epsilon_{k+q,\eta_2}) - 1] \\ &\times [F(\epsilon_{p,\eta_1}) - F(\epsilon_{p-q,\eta_3})] \Big). \end{aligned} \tag{4.108}$$

With the parametrisation $F = 1 - 2f$ this becomes

$$\begin{aligned} I_{coll}[f_{\mathbf{k},\eta}] =& -\,\frac{32\pi}{L^2} \sum_{p,q} |V^R_{RPA,q}(\epsilon_{k+q,\eta_1} - \epsilon_{k,\eta})|^2 \, \tilde{B}^{\eta,\eta_2}_{k,k+q} \tilde{B}^{\eta_3,\eta_1}_{p-q,p} \\ &\times \delta(\epsilon_{k,\eta} + \epsilon_{p,\eta_1} - \epsilon_{k+q,\eta_2} - \epsilon_{p-q,\eta_3}) \\ &\times \Big(f(\epsilon_{k+q,\eta_2}) f(\epsilon_{p-q,\eta_3}) \, [1 - f(\epsilon_{k,\eta})] \, [1 - f(\epsilon_{p,\eta_1})] \\ &- f(\epsilon_{k,\eta}) f(\epsilon_{p,\eta_1}) \, [1 - f(\epsilon_{k+q,\eta_2})] \, [1 - f(\epsilon_{p-q,\eta_3})] \Big). \end{aligned} \tag{4.109}$$

This form of the collision integral could also have been derived from Fermi's golden rule. The only difference is that the interaction is assumed to be screened by density fluctuations.

In the main part of the thesis we will model the collision integral with Fermi's golden rule from the start and assume a short-range interaction in the model. One should keep in mind though that this is a approximation in the lowest order of self energy and the interaction is short-range due to screening.

There is another important point worth mentioning. In the whole calculation so far we have assumed that quasiparticles are well-defined. However, in the previous chapter we argued that one dimensional electrons constitute a prime example of a non-Fermi-liquid and there are no coherent electronic excitations. How can we reconcile those two points? In the case of the HLL the spin degree of freedom is locked to the momentum and we are therefore dealing with effectively spinless particles. The resulting theory is characterized by a line of fixed points in parameter space with K=const. The Luttinger parameter K in turn is specified by g_2 and g_4 which are equal in our case, i.e. $g_2 = g_4 = V_0$, as can be seen from the model for helical fermions in Appendix A. The g_4 term actually vanishes as can be seen by transforming it to real space where it is a local operator (see Eq. (5.65)). However, the g_2 term will change the physics of the system. For now we ignore the g_2 term by setting $g_2 = 0$. This assumption enables us to calculate results using a kinetic equation valid only for well-defined quasiparticles. Furthermore, if we are in the vicinity of the fixed point K=1 the system is approximately conformally invariant which enables us to use arguments of conformal field theory introduced in Sec. 3.1. The effects of finite g_2 will be included in Ch. 7.2 through the renormalization of the coupling constants of the effective model.

5. AC conductivity of a helical Luttinger liquid

In this chapter we derive the AC conductivity for a HLL in a long wire i.e. we assume that the system length L is much larger than the mean free path of electrons. By AC limit we mean the high frequency regime, where the frequency is much larger than the inverse scattering time, i.e. $\omega \gg \tau^{-1}$. τ in turn has to be determined for each process, respectively.

First, we introduce the basic formalism and the model in Sec. 5.1. These will be used to calculate the conductivity corrections in Sec. 5.2 and 5.3. After having identified the most relevant scattering mechanisms we derive effective Hamiltonians describing them in Sec. 5.5 and calculate the emerging corrections to conductivity in Sec. 5.6.

5.1. Formalism and model

5.1.1. Kinetic equation

In equilibrium, non-interacting one-dimensional helical fermions have a linear spectrum $\epsilon_{k,\eta} = \eta k$ and obey the Fermi-Dirac distribution $f^0_{\eta,k} = (1 + \exp\{(\epsilon_{\eta,k} - \mu)/T\})^{-1}$. Away from equilibrium the distribution function $f_{k,\eta}(x)$ has to be determined as the solution of a kinetic equation:

$$\partial_t f_{k,\eta}(x) + v_{k,\eta}\partial_x f_{k,\eta}(x) - eE\partial_k f_{k,\eta}(x) = I_k[f_\eta], \tag{5.1}$$

where $I_k[f_\eta]$ denotes the collision integral. The system we consider is an infinite, homogeneous wire, so we can neglect the spatial dependance of the distribution function. Furthermore, the external electric field E is supposed to be weak. In this case the distribution function will not differ much from the equilibrium Fermi-Dirac distribution and we can expand it as $f_\eta \approx f^0_\eta + f^1_\eta$. Moreover, it will prove useful to parametrise the deviation f^1 with another function ψ as

$$f^1_{\eta,k} \equiv f^0_{\eta,k}(1 - f^0_{\eta,k})\psi_{\eta,k}(t). \tag{5.2}$$

Thus we arrive at the following equation for ψ in the frequency domain:

$$-i\omega\psi_{\eta,k}(\omega) f^0_{\eta,k}(1 - f^0_{\eta,k}) - eE\partial_k f^0_{\eta,k} = I_{\eta,k}[\psi], \tag{5.3}$$

where we already made use of the fact that the collision integral is a linear functional and is annihilated by the Fermi distribution i.e. $I_{\eta,k}[f^0] = 0$.
The equation (5.3) can formally be rewritten into an integral equation for ψ:

$$\psi_{\eta,k}(\omega) = \frac{I_{\eta,k}[\psi]}{(-i\omega)f^0_{\eta,k}(1-f^0_{\eta,k})} - \frac{eE\eta}{(-i\omega)T}, \tag{5.4}$$

where we used the fact that $\partial_k f^0_{\eta,k} = -\eta f^0_{\eta,k}(1-f^0_{\eta,k})/T$.

The collision integral itself is given by

$$I_1^{(1P)}[f] = -\sum_{1'} W_{1,1'}\left[f_1 - f_{1'}\right], \tag{5.5}$$

$$I_1^{(2P)}[f] = -\sum_{2,1',2'} W_{12,1'2'}[f_1 f_2(1-f_{1'})(1-f_{2'}) - f_{1'}f_{2'}(1-f_1)(1-f_2)] \tag{5.6}$$

for one-particle or two-particle scattering processes, respectively. Furthermore, we introduced the joint index $1 \equiv (k_1, \eta_1)$. Next we linearize the expression for the collision integral in ψ:

$$I_1^{(1P)}[\psi] = -\sum_{1'} W_{1,1'}\left[f^0_1(1-f^0_1)\psi_1 - f^0_{1'}(1-f^0_{1'})\psi_{1'}\right], \tag{5.7}$$

$$I_1^{(2P)}[\psi] = -\sum_{2,1',2'} W_{12,1'2'}\left[f^0_1 f^0_2(1-f^0_{1'})(1-f^0_{2'})\left(\psi_1+\psi_2-\psi_{1'}-\psi_{2'}\right)\right]. \tag{5.8}$$

The transition probability $W_{12,1'2'}$ will be calculated by using the generalized Fermi's golden rule in the first Born approximation

$$W_{12,1'2'} = 2\pi N_{\text{imp}} \left|\langle 1'2'|\hat{T}|12\rangle\right|^2 \delta(\epsilon_i - \epsilon_f), \tag{5.9}$$

The energies in the initial and final states are given by $\epsilon_i = \epsilon_1 + \epsilon_2$ and $\epsilon_f = \epsilon_{1'} + \epsilon_{2'}$ and the states $|12\rangle$, $|1'2'\rangle$ are eigenstates of the non-interacting Hamiltonian. The T-matrix is given by the expression:

$$\hat{T} = \left(\hat{H}_{\text{int}} + \hat{H}_{\text{V}}\right) + \left(\hat{H}_{\text{int}} + \hat{H}_{\text{V}}\right)\hat{G}_0\left(\hat{H}_{\text{int}} + \hat{H}_{\text{V}}\right) + \cdots. \tag{5.10}$$

Here the Green's function operator is defined as

$$\hat{G}_0 = \frac{1}{\eta_i k_i - \hat{H}_0 + i0}. \tag{5.11}$$

For a discussion of these formulas see for example Ref. [33].

Some remarks are in order. First, we assume that the interaction strength V_0 and the impurity potential U_0 are weak and therefore we can restrict our calculation to the lowest orders of the T-matrix. Second, we employ the first Born approximation. Therefore, the transition probability in Eq. (5.9) is given by the number of scattering centers N_{imp} times the single particle T-matrix. This approximation neglects multiple scattering off impurities and is valid as long as the inverse impurity scattering time is much smaller than the typical electronic energy i.e. $k_F \gg \tau^{-1}$.

Continuing with our formal manipulations let us rename:

$$C_{1,1'} \equiv \left[f^0_1(1-f^0_1)\psi_1 - f^0_{1'}(1-f^0_{1'})\psi_{1'}\right]\delta(\eta_1 k_1 - \eta_{1'}k_{1'}), \tag{5.12}$$

$$C_{12,1'2'} \equiv \left[f^0_1 f^0_2(1-f^0_{1'})(1-f^0_{2'})\left(\psi_1+\psi_2-\psi_{1'}-\psi_{2'}\right)\right]\delta(\eta_1 k_1+\eta_2 k_2-\eta_{1'}k_{1'}-\eta_{2'}k_{2'}). \tag{5.13}$$

Thus the final forms of the collision integrals (5.7) and (5.8) reads

$$I_1^{(1P)}[\psi] = -2\pi N_{\text{imp}} \sum_{1'} C_{1,1'} \left|\langle 1'|\hat{T}|1\rangle\right|^2, \tag{5.14}$$

$$I_1^{(2P)}[\psi] = -2\pi N_{\text{imp}} \sum_{2,1',2'} C_{12,1'2'} \left|\langle 1'2'|\hat{T}|12\rangle\right|^2. \tag{5.15}$$

After having defined the necessary objects for our calculations we summarize their symmetry properties in the following:

- It is trivial to show that the Fermi-Dirac distribution obeys $f^0_{k\eta} = f^0_{-k\bar{\eta}}$.
- In the absence of scattering i.e. when the collision integral vanishes, the symmetries of the solution ψ of Eq. (5.3) are determined by the driving term $eE\eta f^0_{\eta,k}(1-f^0_{\eta,k})/T$ and therefore:

  $$\psi_{k,\eta} = -\psi_{-k,\bar{\eta}}. \tag{5.16}$$

  It remains true that there exist solutions with this symmetry even in the presence of relaxation inducing processes (although we cannot exclude the possibility that there are solutions without this symmetry). This can be shown explicitely by considering concrete collision integrals in Eq. (5.4).
- The object $C_{12,1'2'}$ is invariant under exchange of the first and second two arguments e.g. $C_{12,1'2'} = C_{21,1'2'}$, which is obvious from its definition. Furthermore, under the assumption of (5.16) it is straightforward to show that $C_{12,1'2'} = -C_{-1,-2,-1',-2'}$ where $-1 \equiv (-k_1, \bar{\eta}_1)$.

5.1.2. Hamiltonian

We use the model for one-dimensional helical fermions derived in Sec. 2.2.4 and Appendix A:

$$\hat{H}_0 = \sum_{k,\eta} \eta\, k\ \hat{\psi}^\dagger_{\eta,k}\hat{\psi}_{\eta,k}, \quad \hat{H}_{\text{int}} = \sum_{i=1}^{5} \hat{H}_i, \tag{5.17}$$

$$\hat{H}_1 = \frac{V_0}{k_0^4 L} \sum_{k,p,q,\eta} \left(k^2-(k-q)^2\right)\left(p^2-(p+q)^2\right) \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\bar{\eta},p}\hat{\psi}_{\bar{\eta},k-q}\hat{\psi}_{\eta,p+q}, \tag{5.18}$$

$$\hat{H}_2 = \frac{V_0}{L} \sum_{k,p,q,\eta} \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\bar{\eta},p}\hat{\psi}_{\bar{\eta},p+q}\hat{\psi}_{\eta,k-q}, \tag{5.19}$$

$$\hat{H}_3 = \frac{V_0}{k_0^4 L} \sum_{k,p,q,\eta} \left(k^2-(k-q)^2\right)\left(p^2-(p+q)^2\right) \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\eta,p}\hat{\psi}_{\bar{\eta},p+q}\hat{\psi}_{\bar{\eta},k-q}, \tag{5.20}$$

$$\hat{H}_4 = \frac{V_0}{L} \sum_{k,p,q,\eta} \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\eta,p}\hat{\psi}_{\eta,p+q}\hat{\psi}_{\eta,k-q}, \tag{5.21}$$

$$\hat{H}_5 = -\frac{V_0}{k_0^2 L} \sum_{k,p,q,\eta} (k^2-p^2)\eta\hat{\psi}^\dagger_{\eta,k+q}\hat{\psi}^\dagger_{\bar{\eta},p-q}\hat{\psi}_{\eta,p}\hat{\psi}_{\eta,k} + h.c., \tag{5.22}$$

$$\hat{H}_{\text{imp}} = \frac{U_0}{L} \sum_{k,p,\eta} \left(\hat{\psi}^\dagger_{\eta,k}\hat{\psi}_{\eta,p} + \eta\frac{k^2-p^2}{k_0^2}\hat{\psi}^\dagger_{\eta,k}\hat{\psi}_{\bar{\eta},p}\right). \tag{5.23}$$

5.1.3. Formula for AC conductivity

If the electronic distribution function $f_{k,\eta}$ is known, we can calculate the conductivity in our semiclassical approximation as

$$\sigma = \frac{(-e)}{EL}\sum_{k,\eta} v_{k,\eta} f_{k,\eta} \overset{(5.2)}{=} \frac{(-e)}{EL}\sum_{k,\eta} \eta f^0_{k,\eta}(1-f^0_{k,\eta})\psi_{k,\eta} \tag{5.24}$$

$$\overset{(5.4)}{=} \frac{(-e)}{EL(-i\omega)}\sum_{k,\eta}\eta I_{k,\eta}[\psi] + \frac{2e^2}{h}\frac{1}{(-i\omega)}, \tag{5.25}$$

where we used that the velocity of a particle of chirality η is $v_{k,\eta} = \eta$.[1] In the AC case we have $\frac{eE\eta}{(-i\omega)T} \ll 1$ which allows us to solve the integral equation (5.4) by iteration:

$$\psi^{(0)}_{\eta,k} \equiv -\frac{eE\eta}{(-i\omega)T}, \tag{5.26}$$

$$\psi^{(n+1)}_{\eta,k} = \frac{I_{\eta,k}[\psi^{(n)}]}{(-i\omega)f^0_{\eta,k}(1-f^0_{\eta,k})} + \psi^{(n)}_{\eta,k}, \quad n \in \mathbb{N}. \tag{5.27}$$

Here we take into account only the zeroth order which leads to the conductivity

$$\sigma_{\mathrm{AC}} = \underbrace{\frac{2e^2}{h}\frac{1}{(-i\omega)}}_{\sigma^{(0)}} + \underbrace{\frac{(-e)}{EL(-i\omega)}\sum_{k,\eta}\eta I_{k,\eta}[\psi^{(0)}]}_{\sigma^{(1)}}. \tag{5.28}$$

Before calculating the conductivity for the HLL let us gain some more intuition about transport properties of electronic systems by deriving the Drude conductivity from classical arguments. While the arguments themselves are actually wrong, since electrons are considered to be a classical gas of charged carriers, the result is nonetheless correct and the arguments themselves provide a nice physical picture.

In the presence of an electric field a conduction electron will feel two different forces: the force of the electric field $-eE$ and a dissipative friction force $-\frac{m}{\tau}\dot{x}$ inhibiting its free acceleration. The damping could for example originate from scattering off static impurities with the scattering rate τ. The dynamics of the electron are then described by the equation of motion $m\ddot{x} = -eE - \frac{m}{\tau}\dot{x}$, or in Fourier representation $-im\omega v(\omega) = eE(\omega) - \frac{m}{\tau}v(\omega)$. Solving for v we find the current density $j = -nev$ and the conductivity given by $j(\omega) = \sigma(\omega)E(\omega)$ as

$$\sigma(\omega) = \frac{ne^2}{m}\frac{1}{\frac{1}{\tau}-i\omega} = \frac{e^2 v_F}{h}\frac{1}{\frac{1}{\tau}-i\omega}. \tag{5.29}$$

In the second equality we used the fact that for one-dimensional electrons the density is given by $n = \frac{1}{a_0} = \frac{k_F}{2\pi} = \frac{k_F}{h}$ and the Fermi momentum is $k_F = mv_F \Leftrightarrow m = k_F/v_F$. Furthermore, we momentarily reintroduced $\hbar$ and v_F.

The AC limit is defined as the regime $\omega \gg \tau^{-1}$. There the conductivity is determined by the ballistic motion of electrons under the force of the electric field and is given by: $\sigma(\omega) \approx \frac{e^2 v_F}{h}\frac{1}{(-i\omega)}$. Our formula for conductivity (5.28) therefore yields the first two terms in a expansion in $(\omega\tau)^{-1}$ (the additional factor of two comes from the edge channel degeneracy). Conversely, in the diffusive regime where $\omega \ll \tau^{-1}$, the motion of electrons is determined by consecutive scattering events and in particular in the DC limit, $\omega \to 0$, we arrive at the Drude formula $\sigma = e^2 v_F \tau/h$, i.e. the electric field drives a steady current whose value is limited by the rate of scattering processes.

Let us now return to the problem of calculating the AC conductivity of a system of helical fermions. We will first tackle this problem in perturbation theory in orders of the T-matrix.

[1] Note that $v_F \equiv 1$.

5.2. First order of the T-matrix

In the absence of any scattering mechanism the AC conductivity in Eq. (5.28) is given by twice the conductance quantum $\sigma^{(0)} = 2e^2/h(-i\omega)$. However, now we want to study how the quantized conductivity is affected by the influence of interactions and impurity scattering, respectively. In order to calculate these corrections, we have to evaluate matrix elements like $\langle 1'2'|\hat{H}_{int}|12\rangle$ in the first order of the T-matrix in Eq. (5.15). To give an idea how the calculations are carried out we derive the contribution of the g_5 term explicitely. The corresponding Hamiltonian is given by Eq. (5.22). First we calculate the necessary matrix elements by using the fermionic anticommutation relations (3.34) :

$$A^{1'2'12}_{kpq\eta} \equiv \langle 1'2'|\psi^\dagger_{\eta,k+q}\psi^\dagger_{\bar{\eta},p-q}\psi_{\eta,p}\psi_{\eta,k}|12\rangle \tag{5.30}$$

$$= \langle 0|\psi_{1'}\psi_{2'}\psi^\dagger_{\eta,k+q}\psi^\dagger_{\bar{\eta},p-q}\psi_{\eta,p}\psi_{\eta,k}\psi^\dagger_1\psi^\dagger_2|0\rangle \tag{5.31}$$

$$= \delta_{\eta_1\eta}\delta_{\eta_2\eta}\left(\delta_{k_1k}\delta_{k_2p} - (1 \leftrightarrow 2)\right)\left(\delta_{\eta_{1'}\bar{\eta}}\delta_{\eta_{2'}\eta}\delta_{k_{1'},p-q}\delta_{k_{2'},k+q} - (1' \leftrightarrow 2')\right), \tag{5.32}$$

$$B^{1'2'12}_{kpq\eta} \equiv \langle 1'2'|\psi^\dagger_{\eta,k}\psi^\dagger_{\eta,p}\psi_{\bar{\eta},p-q}\psi_{\eta,k+q}|12\rangle \tag{5.33}$$

$$= \langle 0|\psi_{1'}\psi_{2'}\psi^\dagger_{\eta,k}\psi^\dagger_{\eta,p}\psi_{\bar{\eta},p-q}\psi_{\eta,k+q}\psi^\dagger_1\psi^\dagger_2|0\rangle \tag{5.34}$$

$$= \delta_{\eta_{1'}\eta}\delta_{\eta_{2'}\eta}\left(\delta_{k_{1'}p}\delta_{k_{2'}k} - (1' \leftrightarrow 2')\right)\left(\delta_{\eta_1\eta}\delta_{\eta_2\bar{\eta}}\delta_{k_1,k+q}\delta_{k_2,p-q} - (1 \leftrightarrow 2)\right). \tag{5.35}$$

To get the complete expression we include the sum over momenta, chiralities and the momentum dependances:

$$A^{121'2'} = -\frac{V_0}{Lk_0^2}\sum_{k,p,q,\eta}\eta A^{121'2'}_{kpq\eta}(k^2 - p^2) \tag{5.36}$$

$$= -\frac{2V_0}{Lk_0^2}\left(\sum_\eta \eta\delta_{\eta_1\eta}\delta_{\eta_2\eta}\left(\delta_{\eta_{1'}\bar{\eta}}\delta_{\eta_{2'}\eta} - (1' \leftrightarrow 2')\right)\right)\delta_{k_{1'}+k_{2'},k_1+k_2}(k_1^2 - k_2^2), \tag{5.37}$$

$$B^{121'2'} = -\frac{V_0}{Lk_0^2}\sum_{k,p,q,\eta}\eta B^{121'2'}_{kpq\eta}(k^2 - p^2) \tag{5.38}$$

$$= -\frac{2V_0}{Lk_0^2}\left(\sum_\eta \eta\delta_{\eta_{1'}\eta}\delta_{\eta_{2'}\eta}\left(\delta_{\eta_1\bar{\eta}}\delta_{\eta_2\eta} - (1 \leftrightarrow 2)\right)\right)\delta_{k_{1'}+k_{2'},k_1+k_2}(k_{1'}^2 - k_{2'}^2). \tag{5.39}$$

Thus the transition probability in Fermi's golden rule is given by

$$\left|\langle 1'2'|\hat{H}_5|12\rangle\right|^2 = \left|A^{121'2'} + B^{121'2'}\right|^2. \tag{5.40}$$

When calculating the square we get $4 \times 4 = 16$ terms. However, due to the chirality factors only 4 out of these are nonvanishing if summed over external chiralities. Let us clarify this by an example:

$$\sum_{\eta,\eta'}\sum_{\eta_2,\eta_{1'},\eta_{2'}} C_{12,1'2'}\eta\eta'\delta_{\eta_1\eta}\delta_{\eta_2\eta}\delta_{\eta_{1'}\bar{\eta}}\delta_{\eta_{2'}\eta}\delta_{\eta_1\eta'}\delta_{\eta_2\eta'}\delta_{\eta_{1'}\bar{\eta}'}\delta_{\eta_{2'}\eta'} \tag{5.41}$$

$$= C_{(k_1,\eta_1)(k_2,\eta_1),(k_{1'},\bar{\eta}_1)(k_{2'},\eta_1)}. \tag{5.42}$$

However,

$$\sum_{\eta,\eta'}\sum_{\eta_2,\eta_{1'},\eta_{2'}} C_{12,1'2'}\eta\eta'\delta_{\eta_1\eta}\delta_{\eta_2\eta}\delta_{\eta_{1'}\bar{\eta}}\delta_{\eta_{2'}\eta}\delta_{\eta_1\eta'}\delta_{\eta_2\bar{\eta}'}\delta_{\eta_{1'}\eta'}\delta_{\eta_{2'}\eta'} \tag{5.43}$$

$$= \sum_{\eta,\eta'} C_{(k_1,\eta_1)(k_2,\eta),(k_{1'},\bar{\eta})(k_{2'},\eta)}\eta\eta'\delta_{\eta'\bar{\eta}}\delta_{\eta'\eta}\delta_{\eta_1\eta}\delta_{\eta_1\eta'} = 0. \tag{5.44}$$

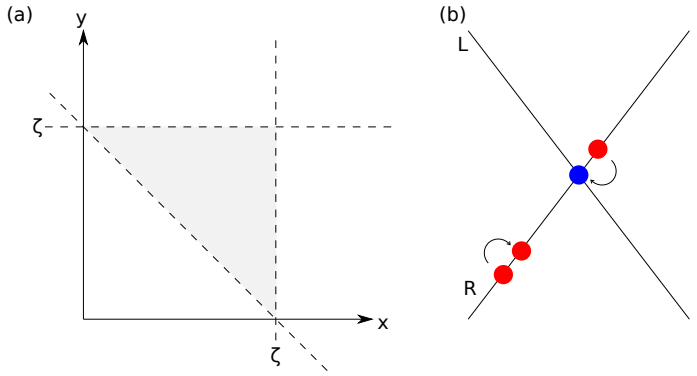

Figure 5.1.: (a) Integration region in the limit $\zeta \gg 1$ (grey area) and (b) spectrum showing a possible g_5 process at low temperatures. A right mover (red) scatters into a left moving state (blue) at zero momentum.

Therefore, only chirality factors that get multiplied by themselves survive in the expression for the collision integral. Substituting our result for the matrix elements into the collision integral Eq. (5.15) and using the symmetry properties of C we get

$$\begin{aligned} I_{k_1,\eta_1}[\psi] = -8\pi \left(\frac{V_0}{k_0^2 L}\right)^2 \sum_{k_2,k_{1'},k_{2'}} & [2(k_1^2-k_2^2)^2 C_{(k_1,\eta_1)(k_2,\eta_1),(k_{1'},\bar{\eta}_1)(k_{2'},\eta_1)} \\ & + (k_{1'}^2-k_{2'}^2)^2 (C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\eta_1)(k_{2'},\eta_1)} + C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\bar{\eta}_1)(k_{2'},\bar{\eta}_1)})]. \end{aligned} \tag{5.45}$$

Since we want to calculate conductivity by using Eq. (5.28) it turns out to be more useful to calculate $\sum_1 \eta_1 I_1[\psi]$ instead of $I_1[\psi]$. Therefore, we have to evaluate

$$\begin{aligned} \sum_{\eta_1,k_1} \eta_1 I_{k_1,\eta_1}[\psi^{(0)}] = & -32\pi \left(\frac{V_0}{k_0^2 L}\right)^2 \sum_{\{k\}} (k_1^2-k_2^2)^2 C_{(k_1,R)(k_2,R),(k_{1'},L)(k_{2'},R)} \\ & \times \delta_{k_1+k_2,k_{1'}+k_{2'}} \end{aligned} \tag{5.46}$$

$$= \frac{8}{\pi} \frac{eEL}{(-i\omega)h} \left(\frac{V_0}{k_0^2}\right)^2 T^5 f(\zeta). \tag{5.47}$$

Here we made use of the symmetry properties of C as well as the fact that we sum over all momenta $\{k\} \equiv k_1, k_2, k_{1'}, k_{2'}$, to cancel all but one factor of C. Next we used the momentum and energy conserving delta functions and substituted $k \to x = k/T$ to make the resulting integral dimensionless. This yields the conductivity correction

$$\sigma^{(1)} = -\frac{8}{\pi} \frac{e^2}{(-i\omega)^2 h} \left(\frac{V_0}{k_0^2}\right)^2 T^5 f(\zeta), \tag{5.48}$$

$$f(\zeta) = \int \mathrm{d}x\mathrm{d}y \underbrace{n(x-\zeta)n(y-\zeta)(1-n(x+y-\zeta))}_{t_\zeta(x,y)}(1-n(-\zeta))(x^2-y^2)^2 \tag{5.49}$$

where $\zeta = k_F/T$ is the Fermi energy measured in units of temperature and $n(x) = (1+e^x)^{-1}$ is the Fermi distribution.

The integral $f(\zeta)$ can be calculated approximately in some regimes. If the Fermi energy k_F is much larger than the temperature i.e. $\zeta \gg 1$ the thermal factor $t_\zeta(x,y)$ is always exponentially suppressed except in the region $x < \zeta$, $\zeta - x < y < \zeta$ where $t_\zeta(x,y) \approx 1$. This integration region is shown in Fig. 5.1. Therefore, we can approximate

$$f(\zeta) \approx \int_0^\zeta \mathrm{d}x \int_{\zeta-x}^\zeta \mathrm{d}y\, e^{-\zeta}(x^2-y^2)^2 = \frac{11}{90}\zeta^6 e^{-\zeta}, \quad \text{for } \zeta \gg 1. \tag{5.50}$$

The resulting correction to conductivity in the regime $k_F >> T$ thus reads

$$\sigma^{(1)} = -\frac{22}{45\pi}\frac{2e^2}{(-i\omega)^2 h}\left(\frac{V_0 k_F^3}{k_0^2}\right)^2 \frac{e^{-k_F/T}}{T}. \tag{5.51}$$

The correction has the form one would expect from Drude's law in the AC limit with the scattering time given by

$$\tau^{(\mathrm{AC})}_{g5,\zeta\gg 1} = \frac{45\pi}{22}\left(\frac{k_0^2}{V_0 k_F^3}\right)^2 T e^{k_F/T}. \tag{5.52}$$

We further notice that the contribution to the conductivity in Eq. (5.51) is thermally activated since energy and momentum conservation constrict the momenta in the operator (5.22) to $p = q$. Hence, one of the particles in the final state has to be created at zero momentum deep within the filled Fermi sea (see Fig. 5.1) which leads to the exponential supression in Eq. (5.51).

Conversely, in the regime $\zeta \ll 1$ we can expand $f(\zeta)$ in a Taylor series around $\zeta = 0$ and evaluate the arising integrals numerically to get

$$\frac{8}{\pi} f(\zeta) \approx 306.02 + 26.2\zeta^2, \quad \text{for } \zeta \ll 1. \tag{5.53}$$

Therefore, in this regime the process leads to power law corrections,

$$\sigma^{(1)} = -153.01\frac{2e^2}{(-i\omega)^2 h}\left(\frac{V_0}{k_0^2}\right)^2 T^5. \tag{5.54}$$

Here the corresponding scattering time is

$$\tau^{(\mathrm{AC})}_{g5,\zeta\ll 1} = 6.5\times 10^{-3}\left(\frac{k_0^2}{V_0}\right)^2 T^{-5} \tag{5.55}$$

Let us adress one important point: how is it possible that interactions that conserve momentum, such as the g_5 term, lead to current relaxation? This is suprising since in conventional Fermi Liquids translational invariance implies momentum conservation and entails the persistence of currents in the absence of momentum nonconserving interactions such as impurity scattering. However, as we discussed in Sec. 3.2.2, the current of a one-dimensional electron system is determined by the number of left and right movers and momentum conservation does not imply current conservation. The current relaxation arises from the scattering of right to left movers or vice versa. Therefore, only scattering processes that change the total number of left and right movers can lead to a finite conductivity. This is true even if the processes conserve momentum.[2]

Lastly, let us discuss the effect of the other interaction processes as well as impurity scattering. Interestingly, we find that none of these scattering mechanism give rise to finite conductivity corrections in the lowest order. Physically, it is clear that g_1,g_2 and g_4 processes will not affect the current since none of them change the number of left and right movers. The same argument is true for forward scattering off the impurity. However, the fact that g_3 processes do not affect the current either, is nontrivial. Indeed, one has to add all processes of this type together to show that they cancel.

[2] If this seems strange to you, remember that any process that changes the overall number of left and right movers was an Umklapp term in the original lattice problem. The fact that it conserves momentum emerges only in the continuum limit.

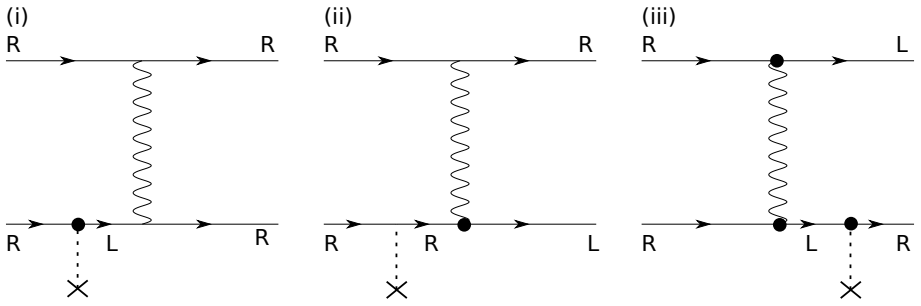

Figure 5.2.: Possible classes of processes in the second order of the T-matrix. Fat vertices denote chirality changes and dashed lines symbolize impurity scattering. The chirality changes can be due to (i) impurities, (ii) interaction or (iii) both.

This is shown explicitely in the calculation in Appendix B. Finally, let us discuss why backscattering off the impurity does not affect transport. These processes are described by the Hamiltonian:

$$\hat{H}_{\mathrm{imp},b} = \frac{U_0}{L}\sum_{k,p,\eta}\eta\frac{k^2-p^2}{k_0^2}\hat{\psi}^\dagger_{\eta,k}\hat{\psi}_{\bar{\eta},p}. \tag{5.56}$$

Energy conservation demands $\eta k = \bar{\eta}p \Leftrightarrow k = -p$ in which case the momentum factor in front vanishes. Therefore, the HLL is indeed topologically protected against single particle scattering as we argued in section 2.2.4.

5.3. Second order of the T-matrix

Having discussed how interaction and impurities respectively affect transport, we now want to investigate combined effects in the next order of the T-matrix. In the second order of our expansion we have the terms:

$$\begin{aligned}\langle 1'2'|\hat{T}|12\rangle = {}& \langle 1'2'|\hat{H}_{\mathrm{int}}\hat{G}_0\hat{H}_{\mathrm{int}}|12\rangle + \langle 1'2'|\hat{H}_{\mathrm{imp}}\hat{G}_0\hat{H}_{\mathrm{int}}|12\rangle \\ & + \langle 1'2'|\hat{H}_{\mathrm{int}}\hat{G}_0\hat{H}_{\mathrm{imp}}|12\rangle + \langle 1'2'|\hat{H}_{\mathrm{imp}}\hat{G}_0\hat{H}_{\mathrm{imp}}|12\rangle.\end{aligned} \tag{5.57}$$

Physically, it is clear that the term with two consecutive impurity scatterings vanishes because of the topological protection. Furthermore, we will neglect processes with two consecutive interaction terms by arguing that they are higher order in V_0. We do this by setting the matrix elements for both processes to zero by hand. Note that these processes would not mix with the others when taking the absolute square since they have different chirality structure due to varied virtual processes. The leading correction to conductivity in this order will therefore arise from combined effects of disorder and interaction. However, only the processes that change the total number of right or left movers can affect current and we are therefore left with three types of possible processes (see Fig. 5.2). In order to calculate the corrections induced by these scattering mechanisms we have to take into account all contributions of the different types. Since the calculations are rather lenghty they are discussed in Appendix C. We find that the leading contribution in the limit $\zeta \gg 1$ comes from combined processes of g_3 and backscattering off an impurity and yield

$$\sigma^{(1)} = -\frac{2e^2}{(-i\omega)^2 h}\kappa\left(\frac{U_0V_0}{k_0^4}\right)^2 n_{\mathrm{imp}}k_F^8T^4, \tag{5.58}$$

where $\kappa = 8^4\times 10.5$ is a numerical factor. These processes are similar to pure g_5 interaction in the sense that they change only the chirality of one particle. Unlike the correction due to interaction, Eq. (5.51), the result for the combined process Eq. (5.58) is not exponentially

suppressed in the limit $k_F \gg T$, though. The exponential suppression is due to the fact that momentum and energy conservation force one of the particles to be at $k = 0$ deep within the Fermi sea. However, if we include impurities, momentum conservation will be broken and the phase space requirements for the process are relaxed. The result is that combined processes of interaction and disorder give the leading order correction to conductivity in the low temperature limit that scale as T^4.

5.4. Comparison with known results

The authors in Ref. [11] studied the conductance of a short wire of lentgh L coupled to external leads held at the same temperature T, but at slightly different chemical potentials $\mu_L = V/2$ and $\mu_R = -V/2$. Here "short" means that the system length is much smaller than the mean free path and transport can be assumed to be ballistic. They calculated the conductance corrections due to interaction and impurities in different regimes of ζ. We can compare our results for the long wire with their calculation if we assume that the frequency of the external electric field ω induces an effective length scale L_{eff} for electrons. Formally we substitute $(-i\omega) \to L_{\text{eff}}^{-1}$ in our results (5.51), (5.54) and (5.58) and then calculate the conductance as $G = \sigma/L_{\text{eff}}$. We find that our results are in agreement with the ones in Ref. [11] and therefore the AC conductivity of a long wire of helical fermions corresponds to the DC conductance of a short wire.

5.5. Derivation of an effective Hamiltonian

Since the calculation of higher order corrections in the T-matrix quickly becomes a cumbersome task we would like to derive an effective Hamiltonian describing combined effects of interaction and disorder. To this end we perform an operator product expansion (OPE) of certain interaction and impurity terms in the Hamiltonian. In quantum field theory the OPE is the representation of a product of two operators (at positions x and y, respectively) as a sum of terms, each consisting of an operator, well-defined as $x \to y$, multiplied by a function of $x - y$ that possibly diverges as $x \to y$. The divergence embodies infinite quantum fluctuations that occur when operators approach the same point. For conformal operators O_i we write

$$O_i(x)O_j(y) = \sum_{k=-\infty}^{\infty} C_{ij}^k \frac{O_k(y)}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}, \tag{5.59}$$

where C_{ij}^k are a set of numbers. It should be understood that the OPE is only meaningful within correlation functions. However, this is no drawback since we need the effective Hamiltonian for the calculation of matrix elements only. The fact that these matrix elements are calculated with the non-interacting ground state $|0\rangle$ enables us to use Wick's theorem for evaluating the OPE. To see why the fused operators describe an effective theory consider a generic fixed point action S_0, describing a conformal field theory and a set of perturbations with coupling constants g_i:

$$S[\phi] = S_0[\phi] + \sum_i g_i \int \mathrm{d}\mathbf{x}\, O_i(\mathbf{x}). \tag{5.60}$$

Here ϕ denote a set of fields defining the model and $\mathbf{x}$ are space-time coordinates. The local operators $O_i(\mathbf{x})$ are in priciple functionals of the fields ϕ. A perturbative RG treatment of the model would yield an expression for the effective action of the form

$$e^{-S_{\text{eff}}[\phi]} \approx e^{-S_0[\phi]}\big[1 - \sum_i g_i \int \mathrm{d}\mathbf{x}\, \langle O_i(\mathbf{x})\rangle + \frac{1}{2}\sum_{ij} g_i g_j \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{x}'\, \langle O_i(\mathbf{x}) O_j(\mathbf{x}')\rangle\big]. \tag{5.61}$$

Here the averaging is over high energy degrees of freedom. In the second order of the expansion we can now fuse the operators by the OPE, Eq. (5.59), where we concentrate on the most diverging terms since they have the highest scaling dimension. The result is that the fused operators O_k appear in the effective action under a RG step and therefore define the effective theory we want to derive.

5.5.1. Hamiltonian in real space representation

For the calculation of OPEs it is convenient to change to a real space representation. Therefore, we Fourier transform the smooth fields $\hat{\psi}_{k\eta}$ as

$$\hat{\psi}_{k\eta} = \frac{1}{\sqrt{L}} \int \mathrm{d}x\, e^{-ikx} e^{-ik_F \eta x} \hat{\psi}_\eta(x). \tag{5.62}$$

The operator for impurity scattering becomes a local operator at x=0:

$$\hat{H}_{\mathrm{imp},f} = U_0 \sum_\eta (\hat{\psi}^\dagger_\eta \hat{\psi}_\eta), \tag{5.63}$$

$$\hat{H}_{\mathrm{imp},b} = \frac{U_0}{k_0^2} \sum_\eta \eta \Big[\hat{\psi}^\dagger_\eta \partial_x^2 \hat{\psi}_{\bar{\eta}} - \partial_x^2 \hat{\psi}^\dagger_\eta \hat{\psi}_{\bar{\eta}} - 2ik_F \eta \left(\partial_x \hat{\psi}^\dagger_\eta \hat{\psi}_{\bar{\eta}} - \hat{\psi}^\dagger_\eta \partial_x \hat{\psi}_{\bar{\eta}} \right) \Big]. \tag{5.64}$$

The g_4 term is given by

$$\hat{H}_4 = V_0 \sum_\eta \int \mathrm{d}x\, \hat{\psi}^\dagger_\eta(x) \hat{\psi}^\dagger_\eta(x+\epsilon) \hat{\psi}_\eta(x+\epsilon) \hat{\psi}_\eta(x), \tag{5.65}$$

where we introduced $\epsilon > 0$ as a pointsplitting constant. In the limit $\epsilon \to 0$ the operator is purely local and vanishes. Furthermore, the remaining operators read

$$\hat{H}_2 = V_0 \sum_\eta \int \mathrm{d}x\, \hat{\psi}^\dagger_\eta(x) \hat{\psi}^\dagger_{\bar{\eta}}(x) \hat{\psi}_{\bar{\eta}}(x) \hat{\psi}_\eta(x) = 2V_0 \int \mathrm{d}x\, (\hat{\psi}^\dagger_R \hat{\psi}^\dagger_L \hat{\psi}_L \hat{\psi}_R)(x), \tag{5.66}$$

$$\hat{H}_5 = \frac{2V_0}{k_0^2} \sum_\eta \eta \int \mathrm{d}x\, e^{-2ik_F \eta x} \left(\hat{\psi}^\dagger_\eta \hat{\psi}^\dagger_{\bar{\eta}} \partial_x^2 \hat{\psi}_\eta \hat{\psi}_\eta - 2ik_F \eta \hat{\psi}^\dagger_\eta \hat{\psi}^\dagger_{\bar{\eta}} \partial_x \hat{\psi}_\eta \hat{\psi}_\eta \right) + h.c.\,, \tag{5.67}$$

$$\begin{aligned} \hat{H}_3 = &-2 \frac{V_0}{k_0^4} \sum_\eta \int \mathrm{d}x\, e^{4i\eta k_F x} \\ &\times \Big[\hat{\psi}^\dagger_\eta \left(\partial_x^2 \hat{\psi}^\dagger_\eta + 2ik_F \eta \partial_x \hat{\psi}^\dagger_\eta - k_F^2 \hat{\psi}^\dagger_\eta \right) \hat{\psi}_{\bar{\eta}} \left(\partial_x^2 \hat{\psi}_{\bar{\eta}} - 2ik_F \eta \partial_x \hat{\psi}_{\bar{\eta}} - k_F^2 \hat{\psi}_{\bar{\eta}} \right) \Big], \end{aligned} \tag{5.68}$$

$$\begin{aligned} \hat{H}_1 = &\frac{V_0}{k_0^4} \sum_\eta \int \mathrm{d}x \\ &\times \Big[\left(\partial_x^2 \hat{\psi}^\dagger_\eta + 2ik_F \eta \partial_x \hat{\psi}^\dagger_\eta - k_F^2 \hat{\psi}^\dagger_\eta \right) \left(\partial_x^2 \hat{\psi}^\dagger_{\bar{\eta}} + 2ik_F \bar{\eta} \partial_x \hat{\psi}^\dagger_{\bar{\eta}} - k_F^2 \hat{\psi}^\dagger_{\bar{\eta}} \right) \hat{\psi}_{\bar{\eta}} \hat{\psi}_\eta \\ &- \hat{\psi}^\dagger_\eta \left(\partial_x^2 \hat{\psi}^\dagger_{\bar{\eta}} + 2ik_F \bar{\eta} \partial_x \hat{\psi}^\dagger_{\bar{\eta}} - k_F^2 \hat{\psi}^\dagger_{\bar{\eta}} \right) \left(\partial_x^2 \hat{\psi}_{\bar{\eta}} - 2ik_F \bar{\eta} \partial_x \hat{\psi}_{\bar{\eta}} - k_F^2 \hat{\psi}_{\bar{\eta}} \right) \hat{\psi}_\eta \\ &- \left(\partial_x^2 \hat{\psi}^\dagger_\eta + 2ik_F \eta \partial_x \hat{\psi}^\dagger_\eta - k_F^2 \hat{\psi}^\dagger_\eta \right) \hat{\psi}^\dagger_{\bar{\eta}} \hat{\psi}_{\bar{\eta}} \left(\partial_x^2 \hat{\psi}_\eta - 2ik_F \eta \partial_x \hat{\psi}_\eta - k_F^2 \hat{\psi}_\eta \right) \\ &+ \hat{\psi}^\dagger_\eta \hat{\psi}^\dagger_{\bar{\eta}} \left(\partial_x^2 \hat{\psi}_{\bar{\eta}} - 2ik_F \bar{\eta} \partial_x \hat{\psi}_{\bar{\eta}} - k_F^2 \hat{\psi}_{\bar{\eta}} \right) \left(\partial_x^2 \hat{\psi}_\eta - 2ik_F \eta \partial_x \hat{\psi}_\eta - k_F^2 \hat{\psi}_\eta \right) \Big]. \end{aligned} \tag{5.69}$$

Power counting suggests that the most relevant operators are those with least derivatives. If we drop all but the most relevant operators the term describing backscattering off impurities becomes:

$$\hat{H}_{\mathrm{imp},b} \approx \frac{U_0}{k_0^2} \sum_\eta \eta \Big[-2ik_F \eta \left(\partial_x \hat{\psi}^\dagger_\eta \hat{\psi}_{\bar{\eta}} - \hat{\psi}^\dagger_\eta \partial_x \hat{\psi}_{\bar{\eta}} \right) \Big] \tag{5.70}$$

$$= \frac{U_0}{k_0^2} (-2ik_F) \left(\partial_x \hat{\psi}^\dagger_R \hat{\psi}_L - \hat{\psi}^\dagger_R \partial_x \hat{\psi}_L + \partial_x \hat{\psi}^\dagger_L \hat{\psi}_R - \hat{\psi}^\dagger_L \partial_x \hat{\psi}_R \right). \tag{5.71}$$

Now we apply the following gauge transformation:

$$\hat{\psi}_L \to i\hat{\psi}_L \ , \qquad \hat{\psi}_L^\dagger \to -i\hat{\psi}_L^\dagger. \tag{5.72}$$

The Hamiltonian is invariant under this transformation except for the g_5 term. Under this transformation the backscattering off the impurity becomes a term describing backscattering from a Rashba impurity:

$$\hat{H}_R \equiv \hat{H}_{\mathrm{imp},b} = \frac{2k_F U_0}{k_0^2}\left(\partial_x\hat{\psi}_R^\dagger\hat{\psi}_L - \hat{\psi}_R^\dagger\partial_x\hat{\psi}_L + h.c.\right). \tag{5.73}$$

Taking into account only the most relevant terms, the g_5 term is given by

$$\hat{H}_5 \approx \frac{2V_0}{k_0^2}\sum_\eta \eta\int \mathrm{d}x\, e^{-2ik_F\eta x}\left(-2ik_F\eta\hat{\psi}_\eta^\dagger\hat{\psi}_{\bar{\eta}}^\dagger\partial_x\hat{\psi}_\eta\hat{\psi}_\eta\right) + h.c.\,. \tag{5.74}$$

After the gauge transformation (5.72) the operator becomes

$$\hat{H}_5 = -\frac{4k_F V_0}{k_0^2}\int \mathrm{d}x\left(e^{-2ik_F x}\hat{\psi}_R^\dagger\hat{\psi}_L^\dagger\partial_x\hat{\psi}_R\hat{\psi}_R - e^{2ik_F x}\hat{\psi}_L^\dagger\hat{\psi}_R^\dagger\partial_x\hat{\psi}_L\hat{\psi}_L + h.c.\right). \tag{5.75}$$

5.5.2. Inelastic scattering

We will call scattering processes that change the chirality of only one incoming particle inelastic scattering processes, since they are similar to scattering off impurities in this aspect but do not conserve single particle energy. To derive the effective Hamiltonian describing such processes we start by fusing g_2 interaction and Rashba scattering:

$$:\hat{\psi}_R^\dagger\hat{\psi}_L^\dagger\hat{\psi}_L\hat{\psi}_R:\Big|_y : \partial_x\hat{\psi}_R^\dagger\hat{\psi}_L - \hat{\psi}_R^\dagger\partial_x\hat{\psi}_L + \hat{\psi}_L^\dagger\partial_x\hat{\psi}_R - \partial_x\hat{\psi}_L^\dagger\hat{\psi}_R :\Big|_{x=0} \tag{5.76}$$

$$\begin{aligned}= &-\langle\hat{\psi}_{L,y}^\dagger\hat{\psi}_{L,x}\rangle :\hat{\psi}_{R,y}^\dagger\hat{\psi}_{L,y}\hat{\psi}_{R,y}\partial_x\hat{\psi}_R^\dagger: - \langle\hat{\psi}_{R,y}\hat{\psi}_{R,x}^\dagger\rangle :\hat{\psi}_{R,y}^\dagger\hat{\psi}_{L,y}^\dagger\hat{\psi}_{L,y}\partial_x\hat{\psi}_L:\\ &-\langle\hat{\psi}_{L,y}\hat{\psi}_{L,x}^\dagger\rangle :\hat{\psi}_{R,y}^\dagger\hat{\psi}_{L,y}^\dagger\hat{\psi}_{R,y}\partial_x\hat{\psi}_R: - \langle\hat{\psi}_{R,y}^\dagger\hat{\psi}_{R,x}\rangle :\hat{\psi}_{L,y}^\dagger\hat{\psi}_{L,y}\hat{\psi}_{R,y}\partial_x\hat{\psi}_L^\dagger: .\end{aligned} \tag{5.77}$$

Here the dots denote normal ordering and the time ordered expectation values of fermionic operators are:

$$\langle\mathbb{T}\hat{\psi}_R(x_1,\tau_1)\hat{\psi}_R^\dagger(x_2,\tau_2)\rangle = \frac{1}{2\pi}\frac{1}{\tau_1-\tau_2+i(x_2-x_1)}, \tag{5.78}$$

$$\langle\mathbb{T}\hat{\psi}_L(x_1,\tau_1)\hat{\psi}_L^\dagger(x_2,\tau_2)\rangle = \frac{1}{2\pi}\frac{1}{\tau_1-\tau_2+i(x_1-x_2)}. \tag{5.79}$$

For time independant operators the following relations hold:

$$\begin{aligned}\langle\mathbb{T}\hat{\psi}_R(x_1)\hat{\psi}_R^\dagger(x_2)\rangle &= -\langle\mathbb{T}\hat{\psi}_R^\dagger(x_2)\hat{\psi}_R(x_1)\rangle = \langle\mathbb{T}\hat{\psi}_R^\dagger(x_1)\hat{\psi}_R(x_2)\rangle &(5.80)\\ &= -\langle\mathbb{T}\hat{\psi}_L(x_1)\hat{\psi}_L^\dagger(x_2)\rangle\,. &(5.81)\end{aligned}$$

Therefore, we can combine the terms in Eq. (5.77) to get

$$\langle\hat{\psi}_{L,y}\hat{\psi}_{L,x}^\dagger\rangle\left(-\hat{\psi}_R^\dagger\hat{\psi}_L\hat{\psi}_R\partial_x\hat{\psi}_R^\dagger + \hat{\psi}_R^\dagger\hat{\psi}_L^\dagger\hat{\psi}_L\partial_x\hat{\psi}_L + \hat{\psi}_L^\dagger\hat{\psi}_L\hat{\psi}_R\partial_x\hat{\psi}_L^\dagger - \hat{\psi}_R^\dagger\hat{\psi}_L^\dagger\hat{\psi}_R\partial_x\hat{\psi}_R\right). \tag{5.82}$$

If we include the corresponding constants we can write the resulting Hamiltonian as

$$\begin{aligned}\hat{H}_{1P}(x=0) =& \frac{1}{2\pi i}\frac{4U_0V_0k_F}{k_0^2}(-\hat{\psi}_R^\dagger\hat{\psi}_L\hat{\psi}_R\partial_x\hat{\psi}_R^\dagger + \hat{\psi}_R^\dagger\hat{\psi}_L^\dagger\hat{\psi}_L\partial_x\hat{\psi}_L\\ &+\hat{\psi}_L^\dagger\hat{\psi}_L\hat{\psi}_R\partial_x\hat{\psi}_L^\dagger - \hat{\psi}_R^\dagger\hat{\psi}_L^\dagger\hat{\psi}_R\partial_x\hat{\psi}_R) &(5.83)\\ =& g_{1P}\left(\sum_\eta\hat{\psi}_\eta^\dagger\hat{\psi}_{\bar{\eta}}^\dagger\hat{\psi}_\eta(i\partial_x)\hat{\psi}_\eta + h.c.\right). &(5.84)\end{aligned}$$

Here we defined the coupling constant:[3]

$$g_{1P} = \frac{2k_F V_0 U_0}{\pi k_0^2}. \tag{5.85}$$

One can check that the fusion of g_5 and forward scattering gives the same Hamiltonian with the same coupling constant. In principle there are further operators that can be combined to get the 1P scattering Hamiltonian e.g. g_3 combined with backscattering. However, they are higher order in the small parameter k_F/k_0 and will not be discussed here. Lastly, the Fourier transform back to momentum space yields:

$$\hat{H}_{1P} = -\frac{g_{1P}}{L^2} \sum_{k,p,q,q',\eta} k\, \hat{\psi}^\dagger_{q',\eta} \hat{\psi}^\dagger_{q,\bar{\eta}} \hat{\psi}_{p,\eta} \hat{\psi}_{k,\eta} + h.c.\,. \tag{5.86}$$

Note that we neglected further terms with a factor of k_F in $\hat{H}_{1P}$ since they have vanishing matrix elements.

5.5.3. Two particle scattering

Scattering processes that change the chirality of both incoming particles are termed two particle scattering processes. To derive the effective Hamiltonian describing such interactions, we can combine either g_5 or the effective one particle process in Eq. (5.84) with backscattering off the impurity. Calculating the necessary contractions yields:

$$\begin{aligned}
&: \partial_y \hat{\psi}^\dagger_R \hat{\psi}_L - \hat{\psi}^\dagger_R \partial_y \hat{\psi}_L + \hat{\psi}^\dagger_L \partial_y \hat{\psi}_R - \partial_y \hat{\psi}^\dagger_L \hat{\psi}_R :\Big|_{y=0} \\
&\times : \pm \hat{\psi}^\dagger_R \hat{\psi}_L \hat{\psi}_R \partial_x \hat{\psi}^\dagger_R \mp \hat{\psi}^\dagger_R \hat{\psi}^\dagger_L \hat{\psi}_L \partial_x \hat{\psi}_L - \hat{\psi}^\dagger_R \hat{\psi}^\dagger_L \hat{\psi}_R \partial_x \hat{\psi}_R + \hat{\psi}^\dagger_L \hat{\psi}_L \hat{\psi}_R \partial_x \hat{\psi}^\dagger_L :
\end{aligned} \tag{5.87}$$

$$\begin{aligned}
&= \langle \hat{\psi}_{L,y} \hat{\psi}^\dagger_{L,x} \rangle \Big(: \pm \partial_y \hat{\psi}^\dagger_R \hat{\psi}^\dagger_{R,x} \hat{\psi}_{L,x} \partial_x \hat{\psi}_L + \partial_y \hat{\psi}^\dagger_R \hat{\psi}^\dagger_{R,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}_R + \partial_y \hat{\psi}^\dagger_R \hat{\psi}_{L,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}^\dagger_L : \Big) \\
&+ \langle \hat{\psi}^\dagger_{R,y} \hat{\psi}_{R,x} \rangle \Big(: \pm \partial_y \hat{\psi}_L \hat{\psi}^\dagger_{R,x} \hat{\psi}_{L,x} \partial_x \hat{\psi}^\dagger_R - \partial_y \hat{\psi}_L \hat{\psi}^\dagger_{R,x} \hat{\psi}^\dagger_{L,x} \partial_x \hat{\psi}_R + \partial_y \hat{\psi}_L \hat{\psi}^\dagger_{L,x} \hat{\psi}_{L,x} \partial_x \hat{\psi}^\dagger_L : \Big) \\
&+ \langle \hat{\psi}^\dagger_{L,y} \hat{\psi}_{L,x} \rangle \Big(: \pm \partial_y \hat{\psi}_R \hat{\psi}^\dagger_{R,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}^\dagger_R \pm \partial_y \hat{\psi}_R \hat{\psi}^\dagger_{R,x} \hat{\psi}^\dagger_{L,x} \partial_x \hat{\psi}_L + \partial_y \hat{\psi}_R \hat{\psi}^\dagger_{L,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}^\dagger_L : \Big) \\
&- \langle \hat{\psi}_{R,y} \hat{\psi}^\dagger_{R,x} \rangle \Big(: \pm \partial_y \hat{\psi}^\dagger_L \hat{\psi}_{L,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}^\dagger_R \mp \partial_y \hat{\psi}^\dagger_L \hat{\psi}^\dagger_{L,x} \hat{\psi}_{L,x} \partial_x \hat{\psi}_L - \partial_y \hat{\psi}^\dagger_L \hat{\psi}^\dagger_{L,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}_R : \Big) .
\end{aligned} \tag{5.88}$$

Here the upper sign corresponds to the g_5 process while the lower describes the inelastic scattering Hamiltonian. The only operators that can change the current are the ones that change the number of right or left moving particles. Therefore, we are left with:

$$\pm \langle \hat{\psi}_{L,y} \hat{\psi}^\dagger_{L,x} \rangle : \partial_y \hat{\psi}^\dagger_R \hat{\psi}^\dagger_{R,x} \hat{\psi}_{L,x} \partial_x \hat{\psi}_L : \pm \langle \hat{\psi}^\dagger_{R,y} \hat{\psi}_{R,x} \rangle : \partial_y \hat{\psi}_L \hat{\psi}^\dagger_{R,x} \hat{\psi}_{L,x} \partial_x \hat{\psi}^\dagger_R : \tag{5.89}$$

$$+ \langle \hat{\psi}^\dagger_{L,y} \hat{\psi}_{L,x} \rangle : \partial_y \hat{\psi}_R \hat{\psi}^\dagger_{L,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}^\dagger_L : + \langle \hat{\psi}_{R,y} \hat{\psi}^\dagger_{R,x} \rangle : \partial_y \hat{\psi}^\dagger_L \hat{\psi}^\dagger_{L,x} \hat{\psi}_{R,x} \partial_x \hat{\psi}_R : \tag{5.90}$$

Using the symmetry properties in Eq. (5.81) we get

$$2 \langle \hat{\psi}_{L,y} \hat{\psi}^\dagger_{L,x} \rangle \left(\pm \partial_x \hat{\psi}^\dagger_R \hat{\psi}^\dagger_R \hat{\psi}_L \partial_x \hat{\psi}_L + \partial_x \hat{\psi}^\dagger_L \hat{\psi}^\dagger_{L,x} \hat{\psi}_R \partial_x \hat{\psi}_R \right). \tag{5.91}$$

Thus we have the two operators

$$\hat{H}_{2P,1} = g_{2P,1} \left(\hat{\psi}^\dagger_R \partial_x \hat{\psi}^\dagger_R \hat{\psi}_L \partial_x \hat{\psi}_L + h.c. \right), \tag{5.92}$$

$$\hat{H}_{2P,2} = g_{2P,2} \left(-i \hat{\psi}^\dagger_R \partial_x \hat{\psi}^\dagger_R \hat{\psi}_L \partial_x \hat{\psi}_L + h.c. \right), \tag{5.93}$$

[3] Here we choose the convention that this operator comes with a factor $(y-x)^{-1}$ in the RG which determines the sign of the coupling constant.

where we defined the coupling constants

$$g_{2\mathrm{P},1} = -\frac{4k_F^2 U_0^2 V_0}{\pi^2 k_0^4}, \qquad g_{2\mathrm{P},2} = \frac{8k_F^2 U_0 V_0}{\pi k_0^4}. \tag{5.94}$$

If we apply the global gauge transformation $\hat{\psi}_L \to \exp(i\pi/4)\hat{\psi}_L$ to the Hamiltonian $\hat{H}_{2\mathrm{P},2}$ the operator becomes identical to $\hat{H}_{2\mathrm{P},1}$ (except for the coupling constant). Therefore, they describe the same physics and we will always use the form $\hat{H}_{2\mathrm{P}} = \hat{H}_{2\mathrm{P},1}$ from now on. Furthermore, the coupling constant we will be using is $g_{2\mathrm{P}} = g_{2\mathrm{P},2}$, since $g_{2\mathrm{P},1}$ is subleading in U_0.

One can check that the fusion of g_3 and forward scattering also gives rise to the same operator with coupling constant $g_{2\mathrm{P},2}$. To show this, we keep the terms with two derivatives in the Hamiltonian,

$$\hat{H}_3 \approx -8\frac{V_0 k_F^2}{k_0^4} \sum_\eta \int \mathrm{d}x\, e^{4i\eta k_F x} \hat{\psi}_\eta^\dagger \partial_x \hat{\psi}_\eta^\dagger \hat{\psi}_{\bar{\eta}} \partial_x \hat{\psi}_{\bar{\eta}} \tag{5.95}$$

and combine it with forward scattering off the impurity. Other terms are either less relevant or do not contribute to two particle scattering. Finally, we state the form of the generic Hamiltonian $\hat{H}_{2\mathrm{P}}$ in momentum space:

$$\hat{H}_{2\mathrm{P}} = \frac{g_{2\mathrm{P}}}{L^2} \sum_{k,p,q,q',\eta} kq\ \ \hat{\psi}_{\eta,k}^\dagger \hat{\psi}_{\eta,p}^\dagger \hat{\psi}_{\bar{\eta},q} \hat{\psi}_{\bar{\eta},q'}. \tag{5.96}$$

Here we neglected terms with k_F since they have vanishing matrix elements.

The effective Hamiltonians found in this section and in Sec. 5.5.2 have been independantly derived in [12] where the authors obtained the operators from the highest weight state of the corresponding conformal algebra.

5.6. Corrections to AC conductivity due to effective processes

The calculation of the correction can be found in the Appendix D. Here we only state the results. For inelastic scattering we have

$$\sigma_{1\mathrm{P}}^{(1)} = -103.902\frac{32}{\pi^4}\frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} \left(\frac{k_F V_0 U_0}{k_0^2}\right)^2 T^4. \tag{5.97}$$

The power law behavior in temperature is in agreement with our perturbative results in Eq. (5.58). However, the prefactor is different and in particular it contains a lower power in k_F/k_0 as well as U_0 making it more important in our weak coupling analysis. At the moment we do not understand how this discrepancy arises. This will be the subject of further studies.

Lastly, the conductivity corrections due to two-particle scattering processes read

$$\sigma_{2\mathrm{P}}^{(1)} = -1757.97\frac{64}{\pi^4} n_{\mathrm{imp}} \frac{2e^2}{h(-i\omega)^2} \left(\frac{k_F^2 U_0 V_0}{k_0^4}\right)^2 T^6. \tag{5.98}$$

In the low temperature regime they therefore only give subleading corrections. For later reference we state the general scattering times for single and two-particle scattering processes:

$$\tau_{1\mathrm{P}} = 9.62 \times 10^{-3} \frac{\pi^2}{n_{\mathrm{imp}}} \frac{1}{g_{1\mathrm{P}}^2 T^4} \tag{5.99}$$

$$\tau_{2\mathrm{P}} = 5.69 \times 10^{-4} \frac{\pi^2}{n_{\mathrm{imp}}} \frac{1}{g_{2\mathrm{P}}^2 T^6} \tag{5.100}$$

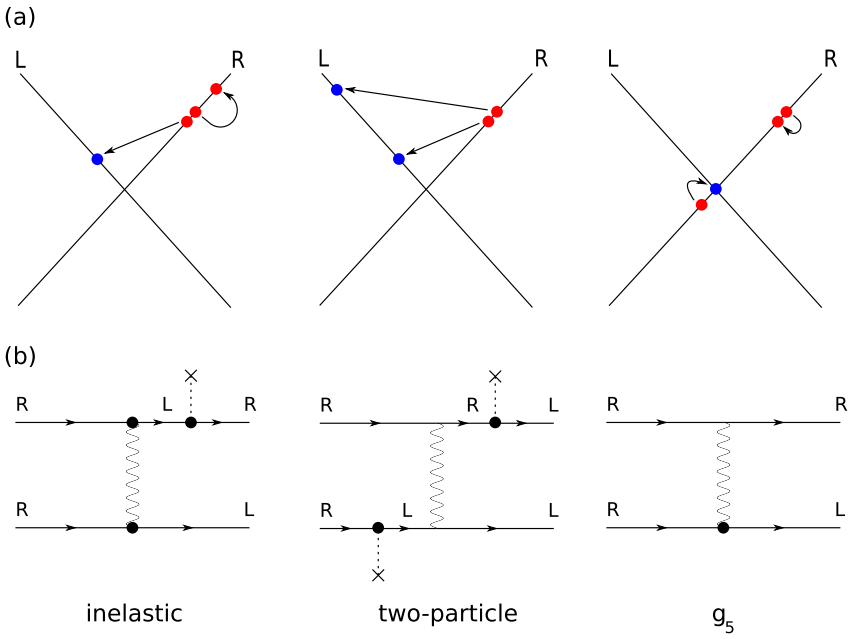

Figure 5.3.: Scattering mechanisms considered in the main text. g_5 processes are due to interaction while inelastic and two-particle processes describe combined effects of disorder and interaction. In (a) we show the corresponding process in the linear spectrum of electrons. (b) depicts possible diagrams describing the mechanisms.

5.7. Summary and results

Let us summarize. We studied the AC conductivity of helical fermions in a long wire. Without any scattering mechanism the conductivity is given by $\sigma^{(0)} = 2e^2/h(-i\omega)$ i.e. the conductance is perfectly quantized. In a clean system there are corrections to this result due to g_5 interaction processes that scale as T^5 at high temperatures (Eq. 5.54) and are exponentially suppressed, as can be shown using phase space arguments, at low temperatures (Eq. 5.51). As expected impurity scattering alone does not influence conductivity because of the topological protection. In the low energy regime we derived the most relevant processes describing combined effects of disorder and impurity scattering with an OPE. The leading order correction comes from inelastic scattering processes and goes as T^4 (Eq. 5.97). We can furthermore map our results to a short wire, with length L_{eff} by the formal substitution $(-i\omega) \to L_{\text{eff}}^{-1}$. This enables us to make predictions about the conductance of such systems as well. The different scattering processes are depicted in Fig. 5.3 and the corresponding corrections to AC conductivity are summarized in Tab. 5.1.

process	Hamiltonian	scattering time τ	
		$T \ll k_F$	$T \gg k_F$
g_5	$\hat{H}_5 = -\frac{V_0}{k_0^2 L} \sum_{k,p,q,\eta} (k^2 - p^2)\eta \left(\hat{\psi}^\dagger_{\eta,k+q} \hat{\psi}^\dagger_{\bar{\eta},p-q} \hat{\psi}_{\eta,p} \hat{\psi}_{\eta,k} + h.c. \right)$	$\frac{45\pi}{22} \left(\frac{k_0^2}{V_0 k_F^3} \right)^2 T e^{k_F/T}$	$6.5 \times 10^{-3} \left(\frac{k_0^2}{V_0} \right)^2 T^{-5}$
1P	$\hat{H}_{1\mathrm{P}} = -\frac{g_{1\mathrm{P}}}{L^2} \sum_{k,p,q,q',\eta} k\, \hat{\psi}^\dagger_{q',\eta} \hat{\psi}^\dagger_{q,\bar{\eta}} \hat{\psi}_{p,\bar{\eta}} \hat{\psi}_{k,\eta} + h.c.$	$9.62 \times 10^{-3} \frac{\pi^2}{n_{\mathrm{imp}}} \frac{1}{g_{1\mathrm{P}}^2} T^{-4}$	
2P	$\hat{H}_{2\mathrm{P}} = \frac{g_{2\mathrm{P}}}{L^2} \sum_{k,p,q,q',\eta} kq\, \hat{\psi}^\dagger_{\eta,k} \hat{\psi}^\dagger_{\bar{\eta},p} \hat{\psi}_{\bar{\eta},q} \hat{\psi}_{\bar{\eta},q'}$	$5.69 \times 10^{-4} \frac{\pi^2}{n_{\mathrm{imp}}} \frac{1}{g_{2\mathrm{P}}^2} T^{-6}$	

Table 5.1.: Most relevant scattering mechanisms and their respective scattering time τ. The correction to AC conductivity can be obtained as: $\sigma^{(1)} = -\frac{2e^2}{(-i\omega)^2 h} \tau^{-1}$. g_5 processes are purely due to electron-electron interaction while inelastic (1P) and two-particle processes are effective Hamiltonians describing combined effects of disorder and interaction. The coupling constants for inelastic and two-particle scattering are $g_{1\mathrm{P}} = \frac{2k_F V_0 U_0}{\pi k_0^2}$ and $g_{2\mathrm{P}} = \frac{8k_F^2 U_0 V_0}{\pi k_0^4}$.

6. DC conductivity of a helical Luttinger liquid

So far we have only considered transport properties in the AC limit where the electrons' motion is ballistic. In this regime we could make predictions about the conductivity in a perturbative expansion in the small parameter $eE/(-i\omega T)$. However, now we want to ask the question what form the conductivity takes in the DC limit. To this end we have to develop a new formalism for finding the corrections since we can not use the same perturbative expansion as before in the limit $\omega \to 0$. Interestingly, even in a conventional LL the calculation of the DC conductivity is a nontrivial problem which was already realized in early publications [41]. It can be shown that even in the presence of an Umklapp term, current is not relaxed and the DC conductivity of a LL is infinite. One needs at least two noncommensurate Umklapp terms to get a finite DC conductivity [42]. In this chapter we discuss the DC conductivity of a HLL in the presence of the most relevant perturbations found in Ch. 5. To this end we first introduce the formalism in Sec. 6.1 and then proceed to analyse g_5, inelastic and two-particle scattering in Secs. 6.2, 6.3 and 6.4, respectively. A summary of our results can be found in Sec. 6.5.

6.1. Formalism

For the calculation of conductivity we use our general expression in Eq. (5.24). The symmetry properties of $\psi_{k,\eta}$, described in Eq. (5.16), allow us to rewrite the conductivity as

$$\sigma_{\mathrm{DC}} = \frac{(-2e)}{Eh} T \int \mathrm{d}x\, n(x-\zeta)(1-n(x-\zeta))\psi_{R,\zeta}(x). \tag{6.1}$$

Here ζ denotes the ratio of Fermi energy and temperature, $\zeta = k_F/T$.

The function $\psi_{k,\eta}$, parametrising deviations from the equilibrium distribution function, has to be determined from the integral equation in Eq. (5.4):

$$\psi_{k,\eta}(\omega) = \frac{I_{\eta,k}[\psi]}{(-i\omega) f^0_{\eta,k}(1-f^0_{\eta,k})} - \frac{eE\eta}{(-i\omega)T}. \tag{6.2}$$

Given a particular collision integral containing the information about the scattering process under consideration, equation (5.4) represents a set of coupled Fredholm integral

equations of the second kind that determine $\psi_{\eta,k}(\omega)$. Luckily, due to the general symmetry, Eq. (5.16), we can decouple the integral equations and are left with the task of solving a single, one-dimensional integral equation. WLOG we will choose to solve for ψ_R.

In the following we use this formalism to calculate the DC conductivity due to g_5, inelastic and two particle processes in different temperature regimes.

6.2. g_5 processes

6.2.1. Derivation of an integral equation for g_5 processes

The collision integral for the g_5 process is calculated in Eq. (5.45). By inserting it into Eq. (5.4) and evaluating the momentum conserving Kronecker delta we obtain the following expression:

$$\begin{aligned}\psi_{k,R} = &- \frac{8\pi}{(-i\omega) f_{kR}(1-f_{kR})} \left(\frac{U_0}{k_0^2 L}\right)^2 \sum_{k_2,k_{1'}} \\ &\times \Big[2(k_1^2-k_2^2)^2 C_{(k_1,R)(k_2,R),(k_{1'},L)(k_1+k_1-k_{1'},R)} \\ &+ (k_{1'}^2-(k_1+k_2-k_{1'})^2)^2\, C_{(k_1,R)(k_2,L),(k_{1'},R)(k_1+k_1-k_{1'},R)} \\ &+ (k_{1'}^2-(k_1+k_2-k_{1'})^2)^2\, C_{(k_1,R)(k_2,L),(k_{1'},L)(k_1+k_1-k_{1'},L)}\Big] \\ &- \frac{eE}{(-i\omega)T}.\end{aligned} \tag{6.3}$$

After evaluating the energy conserving delta functions contained in C and going to a continous momentum representation we arrive at

$$\begin{aligned}\psi_R(k) = &-\frac{8\pi}{(-i\omega)(1-f_{kR})} \left(\frac{U_0}{k_0^2 L}\right)^2 \left(\frac{L}{2\pi}\right)^2 \frac{1}{2} \int \mathrm{d}p \\ &\times [2(k^2-p^2)^2 f^0_{pR}(1-f^0_{0L})(1-f^0_{k+p,R}) \\ &\times (\psi_R(k)+\psi_R(p)-\psi_L(0)-\psi_R(k+p)) \\ &+ ((k-p)^2-p^2)^2 f^0_{0L}(1-f^0_{pR})(1-f^0_{k-p,R}) \\ &\times (\psi_R(k)+\psi_L(0)-\psi_R(p)-\psi_R(k-p))] \\ &+ \int \mathrm{d}q\, ((k+p-q)^2-q^2)^2 f^0_{pL}(1-f^0_{qL})(1-f^0_{k+p-q,L}) \\ &\times \delta(k)(\psi_R(k)+\psi_L(p)-\psi_L(q)-\psi_L(k+p-q)) - \frac{eE}{(-i\omega)T}.\end{aligned} \tag{6.4}$$

Interestingly, in the last term we find a delta function that constricts the external momentum to zero. As discussed in Sec. 5.2, the phenomenon that one particle is forced to have zero momentum arises from the chirality structure of the process combined with the restrictions of energy and momentum conservation. Therefore, we have to consider two cases:
First let $k \neq 0$, then $\delta(k) = 0$ and we are left with

$$\begin{aligned}\psi_R(k) = &-\frac{1}{(-i\omega)(1-f_{kR})\pi} \left(\frac{U_0}{k_0^2}\right)^2 \int \mathrm{d}p\, [2(k^2-p^2)^2 f^0_{pR}(1-f^0_{0L})(1-f^0_{k+p,R}) \\ &\times (\psi_R(k)+\psi_R(p)-\psi_L(0)-\psi_R(k+p)) + ((k-p)^2-p^2)^2 f^0_{0L}(1-f^0_{pR}) \\ &\times (1-f^0_{k-p,R})(\psi_R(k)+\psi_L(0)-\psi_R(p)-\psi_R(k-p))] - \frac{eE}{(-i\omega)T}.\end{aligned} \tag{6.5}$$

If we define:

$$G(k,p) = ((k-p)^2 - p^2)^2 f^0_{0L} \frac{(1-f^0_{pR})}{(1-f_{kR})}(1-f^0_{k-p,R}), \tag{6.6}$$

$$K(k,p) = 2(k^2-p^2)^2 f^0_{pR} \frac{(1-f^0_{0L})}{(1-f_{kR})}(1-f^0_{k+p,R}), \tag{6.7}$$

the terms can be rewritten into a more compact form after some algebra

$$\begin{aligned}\psi_R(k) = -\frac{1}{(-i\omega)(1-f_{kR})\pi}\left(\frac{U_0}{k_0^2}\right)^2 \Big(\psi_R(k)\int \mathrm{d}p\ [K(k,p)+G(k,p)] \\ + \psi_L(0)\int \mathrm{d}p\ [-K(k,p)+G(k,p)] + \psi_R(p)\int \mathrm{d}p\,[K(k,p)-K(k,p-k) \\ - 2G(k,p)]\Big) - \frac{eE}{(-i\omega)T}.\end{aligned} \tag{6.8}$$

For $k=0$, $\delta(k)$ is formally divergent. However, we argue that this divergence is artificial and caused by the assumption of an infinite system. In a finite edge channel this divergence will be regularized by the system length L. Here, we will assume that L is still the largest length scale in the problem and therefore the term containing the system length will be much larger than the other terms in Eq. (6.4). Consequently, the resulting expression for $k=0$ reads

$$\begin{aligned}\psi_{0R} = -\frac{L}{(-i\omega)(1-f_{kR})\pi}\left(\frac{U_0}{k_0^2}\right)^2 \int \mathrm{d}p\mathrm{d}q((k+p-q)^2-q^2)^2 \\ \times f^0_{pL}(1-f^0_{qL})(1-f^0_{p-q,L})(\psi_{0R}+\psi_{pL}-\psi_{qL}-\psi_{p-q,L}) - \frac{eE}{(-i\omega)T}.\end{aligned} \tag{6.9}$$

If we use the symmetry $\psi_{kR} = -\psi_{-k,L}$ and scale all momenta $k \to x = k/T$ to make the integrals dimensionless we arrive at the following form for $x \neq 0$:

$$\begin{aligned}\frac{\psi_{\zeta,\xi,R}(x)}{D} = \frac{\xi A_{\zeta,-}(x)}{1-\xi A_{\zeta,+}(x)}\frac{\psi_{\zeta,\chi,R}(0)}{D} + \frac{\xi}{1-\xi A_{\zeta,+}(x)}\int \mathrm{d}y\, M_\zeta(x,y)\frac{\psi_{\zeta,\xi,R}(y)}{D} \\ - \frac{i}{1-\xi A_{\zeta,+}(x)}.\end{aligned} \tag{6.10}$$

Here we defined the objects

$$\xi = -\frac{1}{\pi(-i\omega)}\left(\frac{U_0}{k_0^2}\right)^2 T^5, \qquad D = \frac{eE}{\omega T}, \tag{6.11}$$

$$G_\zeta(x,y) = \left((x-y)^2-y^2\right)^2 n(-\zeta)(1-n(y-\zeta))\frac{1-n(x-y-\zeta)}{1-n(x-\zeta)}, \tag{6.12}$$

$$K_\zeta(x,y) = 2\left(x^2-y^2\right)^2 n(y-\zeta)(1-n(-\zeta))\frac{1-n(x+y-\zeta)}{1-n(x-\zeta)}, \tag{6.13}$$

$$M_\zeta(x,y) = K_\zeta(x,y) - K_\zeta(x,y-x) - 2G_\zeta(x,y). \tag{6.14}$$

$$A_{\zeta,\pm}(x) = \int \mathrm{d}y\, K_\zeta(x,y) \pm G_\zeta(x,y). \tag{6.15}$$

For $x=0$ we obtain the following integral equation governing $\psi_\zeta(0)$:

$$\begin{aligned}\psi_{\zeta,\chi,R}(0) = \chi\int \mathrm{d}y\mathrm{d}z\, H_\zeta(y,z)\left(\psi_{\zeta,\chi,R}(0) - \psi_{\zeta,\chi,R}(-y) + \psi_{\zeta,\chi,R}(-z) + \psi_{\zeta,\chi,R}(z-y)\right) \\ - iD,\end{aligned} \tag{6.16}$$

where

$$\chi = -\frac{L}{\pi(-i\omega)}\left(\frac{U_0}{k_0^2}\right)^2 T^6, \tag{6.17}$$

$$H_\zeta(y,z) = \left((y-z)^2 - z^2\right)^2 n(-y-\zeta)\frac{1-n(-z-\zeta)}{1-n(-\zeta)}(1-n(-y+z-\zeta)), \tag{6.18}$$

$$H_\zeta = \int \mathrm{d}y\mathrm{d}z\, H_\zeta(y,z). \tag{6.19}$$

Now let us proceed with our manipulations. First we solve Eq. (6.16) for $\psi_{\zeta,\chi,R}(0)$ to get

$$\frac{\psi_{\zeta,\chi,R}(0)}{D} = \frac{\chi}{1-\chi H_\zeta}\int \mathrm{d}y\,(H_{\zeta,1}(y) + H_{\zeta,2}(y) + H_{\zeta,3}(y))\,\frac{\psi_{\zeta,\chi,R}(y)}{D} - \frac{i}{1-\chi H_\zeta} \tag{6.20}$$

$$= \frac{\chi}{1-\chi H_\zeta}\int \mathrm{d}y \tilde{H}_\zeta(y)\psi_{\zeta,\chi,R}(y) - \frac{i}{1-\chi H_\zeta}. \tag{6.21}$$

with

$$H_{\zeta,1}(y) = -\int \mathrm{d}z\, H_\zeta(-y,z), \tag{6.22}$$

$$H_{\zeta,2}(y) = \int \mathrm{d}z\, H_\zeta(z,-y), \tag{6.23}$$

$$H_{\zeta,3}(y) = \int \mathrm{d}z\, H_\zeta(z-y,z), \tag{6.24}$$

$$\tilde{H}_\zeta(y) = H_{\zeta,1}(y) + H_{\zeta,2}(y) + H_{\zeta,3}(y). \tag{6.25}$$

Inserting Eq. (6.21) into the integral equation for $\psi_{\zeta,\xi}(x)$, Eq. (6.10), we obtain

$$\begin{aligned}\frac{\psi_{\zeta,\xi,\chi,R}(x)}{D} =& \frac{\xi A_{\zeta,-}(x)}{1-\xi A_{\zeta,+}(x)}\frac{\chi}{1-\chi H_\zeta}\int \mathrm{d}y \tilde{H}_\zeta(y)\frac{\psi_{\zeta,\xi,\chi,R}(y)}{D} \\ &+ \frac{\xi}{1-\xi A_{\zeta,+}(x)}\int \mathrm{d}y\, M_\zeta(x,y)\frac{\psi_{\zeta,\xi,\chi,R}(y)}{D} - \frac{i}{1-\xi A_{\zeta,+}(x)} \\ &- \frac{\xi A_{\zeta,-}(x)}{1-\xi A_{\zeta,+}(x)}\frac{i}{1-\chi H_\zeta}.\end{aligned} \tag{6.26}$$

Up to this point all manipulations have been exact and Eq. (6.26) fully determines $\psi_{\zeta,\xi,\chi,R}(x)$. In order to solve this integral equation we first have to find expressions for the functions defined through various integrals. This task is tackled in Appendix E.1 where we obtain approximate expressions in different regimes of temperature.

6.2.2. Solution of the integral equation

Unfortunately, the integral equation, Eq. (6.26), cannot be solved exactly. Thus we are forced to look for suitable approximations. Ultimately, we need to know ψ in order to calculate the conductivity in Eq. (6.1). There we have to integrate over ψ together with a function $t_\zeta(x) = n(x-\zeta)(1-n(x-\zeta))$ that is strongly peaked around $x=\zeta$. Therefore, under the assumption that ψ is a smooth function compared to the thermal factor $t_\zeta(x)$ we only need to compute $\psi(x \approx \zeta)$. Especially, we can approximate $M_\zeta(x,y) \approx M_\zeta(\zeta,y)$ in Eq. (6.26). The resulting integral equation is separable and can be solved exactly. We can map the problem of solving an integral equation onto the solution of a system of algebraic

equations by defining

$$\begin{aligned}\frac{\psi_{\zeta,\xi,\chi,R}(x)}{D} =& \frac{\xi A_{\zeta,-}(x)}{1-\xi A_{\zeta,+}(x)}\frac{\chi}{1-\chi H_\zeta}\int \mathrm{d}y\, \tilde{H}_\zeta(y)\frac{\psi_{\zeta,\xi,\chi,R}(y)}{D} \\ &+ \frac{\xi}{1-\xi A_{\zeta,+}(x)}\int \mathrm{d}y\, M_\zeta(\zeta,y)\frac{\psi_{\zeta,\xi,\chi,R}(y)}{D} - \frac{i}{1-\xi A_{\zeta,+}(x)} \\ &- \frac{\xi A_{\zeta,-}(x)}{1-\xi A_{\zeta,+}(x)}\frac{i}{1-\chi H_\zeta} \end{aligned} \tag{6.27}$$

$$\begin{aligned}\equiv& \frac{\xi A_{\zeta,-}(x)}{1-\xi A_{\zeta,+}(x)}\frac{\chi}{1-\chi H_\zeta}C_{1,\zeta,\xi} + \frac{\xi}{1-\xi A_{\zeta,+}(x)}C_{2,\zeta,\xi} - \frac{i}{1-\xi A_{\zeta,+}(x)} \\ &- \frac{\xi A_{\zeta,-}(x)}{1-\xi A_{\zeta,+}(x)}\frac{i}{1-\chi H_\zeta}. \end{aligned} \tag{6.28}$$

Here C_1 and C_2 are given by

$$C_{1,\zeta,\xi,\chi} = \int \mathrm{d}y\, \tilde{H}_\zeta(y)\frac{\psi_{\zeta,\xi,\chi,R}(y)}{D}, \tag{6.29}$$

$$C_{2,\zeta,\xi,\chi} = \int \mathrm{d}y\, M_\zeta(\zeta,y)\frac{\psi_{\zeta,\xi,\chi,R}(y)}{D}. \tag{6.30}$$

Inserting Eq. (6.28) into Eq. (6.29) and (6.30) we find the following set of linear equations determining the C_n[1]:

$$C_{n,\zeta,\xi,\chi} = E_{nm,\zeta,\xi,\chi}C_{m,,\zeta,\xi,\chi} + F_{n,\zeta,\xi,\chi}. \tag{6.31}$$

Here we used the abbreviations

$$E_{11,\zeta,\xi,\chi} = \int \mathrm{d}y\, \tilde{H}_\zeta(y)\frac{\xi A_{\zeta,-}(y)}{1-\xi A_{\zeta,+}(y)}\frac{\chi}{1-\chi H_\zeta}, \tag{6.32}$$

$$E_{12,\zeta,\xi,\chi} = \int \mathrm{d}y\, \tilde{H}_\zeta(y)\frac{\xi}{1-\xi A_{\zeta,+}(y)}, \tag{6.33}$$

$$E_{21,\zeta,\xi,\chi} = \int \mathrm{d}y\, M_\zeta(\zeta,y)\frac{\xi A_{\zeta,-}(y)}{1-\xi A_{\zeta,+}(y)}\frac{\chi}{1-\chi H_\zeta}, \tag{6.34}$$

$$E_{22,\zeta,\xi,\chi} = \int \mathrm{d}y\, M_\zeta(\zeta,y)\frac{\xi}{1-\xi A_{\zeta,+}(y)}, \tag{6.35}$$

$$F_{1,\zeta,\xi,\chi} = -\frac{iE_{11}}{\chi} - \frac{i}{\xi}E_{12}, \tag{6.36}$$

$$F_{2,\zeta,\xi,\chi} = -\frac{iE_{21}}{\chi} - \frac{1}{\xi}E_{22}. \tag{6.37}$$

Notice that we made a nontrivial assumption in this step. The coefficients C_1 and C_2 are determined by an integral over the whole real axis, i.e. we need to know $\psi(y)$ for all $y \in \mathbb{R}$ to calculate them. Eq. (6.28) however, only determines ψ in the region close to ζ. Therefore, we assume that the functions $\tilde{H}_\zeta(y)$ and $M_\zeta(\zeta,y)$ are reasonably well peaked around $y = \zeta$ such that contributions away from this point give a parametrically smaller contribution to the integral in the limit $\zeta \gg 1$.

Until now we have not made use of the fact that we are interested in the DC conductivity i.e. the limit $\omega \to 0$. In this limit the dimensionless parameters ξ and χ diverge and we can make further simplifications. Furthermore, we restrict our calculation to the real part of the conductivity since in the DC limit the conductivity will be a purely real quantity.

[1]Summation convention over latin indices is implied.

For that reason we define $\xi'' = \mathfrak{Im}\,\xi$ and $\chi'' = \mathfrak{Im}\,\chi$. Now the following train of thought holds:

$$|\xi'' \min_x A_{\zeta,+}(x)| \gg 1 \Rightarrow |\xi'' A_{\zeta,+}(x)| \gg 1 \Rightarrow \mathfrak{Re}\,\frac{\xi}{1-\xi A_{\zeta,+}(x)} \approx -\frac{1}{A_{\zeta,+}(x)} \tag{6.38}$$

and analogously for χ and H_ζ. Under the conditions

$$\xi''\,|\min_x A_{\zeta,+}(x)| \gg 1 \quad \wedge \quad \chi'' H_\zeta \gg 1, \tag{6.39}$$

we can approximate the expressions in Eq. (6.32)-(6.35) in the same manner as in Eq. (6.38).

Let us add some remarks. The conditions in Eq. (6.39) constrain the validity of our further calculations to the frequency range

$$\omega^{-1} \gg \left(\frac{k_0^2}{U_0}\right)^2 \frac{1}{k_F^5} \max\{e^\zeta, k_F^{-1}L^{-1}\} = \left(\frac{k_0^2}{U_0}\right)^2 \frac{1}{k_F^5} e^\zeta, \tag{6.40}$$

as can be readily shown by using the definitions of χ and ζ and the fact that $\min_x A_{\zeta,+}(x) \approx A_{\zeta,+}(\zeta) \sim \zeta^5 e^{-\zeta}$ and $H_\zeta \sim \zeta^6$. Here we will assume $e^\zeta > k_F^{-1}L^{-1}$, since the system length is the largest length scale in the problem and therefore in particular bigger than the inverse Fermi momentum. On the other hand, temperature is assumed to be much smaller than the Fermi energy and therefore $\zeta \gg 1$. Summarizing, we assume the following hierarchy of energy scales: $k_F \gg T \gg L^{-1}$.

In principle there are two time scales originating from $\psi(x \neq 0)$ and $\psi(x = 0)$, respectively. In addition to the DC case where ω^{-1} is bigger than both scales (see Eq. (6.40)) and the AC limit where it is smaller, there is also an intermediate regime. This however, is not the focus of our study here and will be left for future work.

We solve the integrals in Eq. (6.32)-(6.35) by estimating the dependance on the parameter ζ and calculating the proportionality constant numerically. The resulting expressions are of course not exact but they represent the leading terms in $\zeta \gg 1$:

$$E_{11,\zeta,\xi} \approx \int \mathrm{d}y\, \tilde{H}_\zeta(y) \frac{A_{\zeta,-}(y)}{A_{\zeta,+}(y)} \frac{1}{H_\zeta} \approx 2.81, \tag{6.41}$$

$$E_{12,\zeta,\xi} \approx -\int \mathrm{d}y\, \tilde{H}_\zeta(y) \frac{1}{A_{\zeta,+}(y)} \approx -0.378 e^\zeta, \tag{6.42}$$

$$E_{21,\zeta,\xi} \approx \int \mathrm{d}y\, M_\zeta(\zeta, y) \frac{A_{\zeta,-}(y)}{A_{\zeta,+}(y)} \frac{1}{H_\zeta} \approx \frac{23.14}{\zeta} e^{-\zeta}, \tag{6.43}$$

$$E_{22,\zeta,\xi} \approx -\int \mathrm{d}y\, M_\zeta(\zeta, y) \frac{1}{A_{\zeta,+}(y)} \approx \frac{6.367}{\zeta}. \tag{6.44}$$

Having calculated the necessary matrix elements we can solve the system of linear equations in Eq. (6.31) by simple matrix inversion:

$$\mathbf{C} = E\mathbf{C} + \mathbf{F} \quad \Leftrightarrow \quad \mathbf{C} = (\mathbb{1} - E)^{-1}\mathbf{F}. \tag{6.45}$$

The result to the leading order in ζ is

$$C_{1,\zeta,\xi} = \frac{26.6\,\xi'' - 2.8\,\zeta\,\xi'' + 0.38 e^\zeta \zeta \chi''}{20.22\,\xi''\chi'' - 1.8\,\zeta\xi''\chi''} \approx -\frac{0.21 e^\zeta}{\xi''} \tag{6.46}$$

$$C_{2,\zeta,\xi} = \frac{-23.1 e^{-\zeta}\xi'' + 20.2\chi''}{20.2\,\xi''\chi'' - 1.8\,\zeta\xi''\chi''} \approx -\frac{11.2}{\zeta\xi''}. \tag{6.47}$$

Noting that $A_{\pm,\zeta}(x \approx \zeta) \sim \zeta^5 e^{-\zeta}$, $H_\zeta \sim \zeta^6$ and using our results from Eq. (6.46) and (6.47) we find from Eq. (6.28) that to leading order $\mathfrak{Re}\,\psi_{\zeta,\xi}(x)$ is given by

$$\mathfrak{Re}\,\frac{\psi_{\zeta,\xi}(x)}{D} \approx \frac{1}{\xi'' A_{\zeta,+}(x)}. \tag{6.48}$$

Therefore, we get an estimate of the real part of conductivity

$$\begin{aligned}
\mathfrak{Re}\,\sigma =& \frac{-2e}{Eh} TD \int \mathrm{d}x\, n(x-\zeta)(1-n(x-\zeta))\,\mathfrak{Re}\,\frac{\psi_{\zeta,\xi}(x)}{D} && (6.49)\\
\approx& \frac{-2e}{Eh} TD \int \mathrm{d}x\, n(x-\zeta)(1-n(x-\zeta))\frac{1}{\xi'' A_{\zeta,+}(x)} && (6.50)\\
=& \frac{2e^2}{h} 0.254 \frac{\pi}{k_F^5}\left(\frac{k_0^2}{U_0}\right)^2 e^{\frac{k_F}{T}}. && (6.51)
\end{aligned}$$

We notice that the conductivity takes the form $\mathfrak{Re}\,\sigma = 2e^2/h \times \tau^{\mathrm{DC}}_{g_5,\zeta\gg 1}$, with the scattering time

$$\tau^{\mathrm{DC}}_{g_5,\zeta\gg 1} = 0.254 \frac{\pi}{k_F^5}\left(\frac{k_0^2}{U_0}\right)^2 e^{\frac{k_F}{T}} \tag{6.52}$$

By comparing this with the corresponding result in the AC case, Eq. (5.52), we see that apart from a small discrepancy in the numerical prefactor $\tau^{\mathrm{DC}}_{g_5,\zeta\gg 1} = \zeta \tau^{\mathrm{AC}}_{g_5,\zeta\gg 1}$. Thus the time between scattering events is much larger in the DC than in the AC case. We can understand this phenomenologically by the following argument. For the g_5 process to take place the state at zero momentum has to be empty. In the AC case the external energy ω is large and can empty the state fast, even at low temperatures. However, in the DC case the state at $k = 0$ can only be emptied by thermal fluctuations and therefore the time between consecutive scattering events is much larger.

Let us now continue by studying the regime in which the temperature is much larger than the Fermi momentum, i.e $\zeta \ll 1$. Then we can expand the matrix elements in Eqs. (6.41)-(6.44) as a power series in ζ. The resulting integrals are calculated numerically using the exact expressions in Appendix E.1. This procedure yields the results

$$\begin{aligned}
E_{11,\zeta,\xi} \approx& \frac{1}{H_\zeta}(258.61 + 153.63\zeta^2), && (6.53)\\
E_{12,\zeta,\xi} \approx& -2.49 - 0.17\zeta^2, && (6.54)\\
E_{21,\zeta,\xi} \approx& -\frac{12.45\zeta^2}{H_\zeta}, && (6.55)\\
E_{22,\zeta,\xi} \approx& 0.023\zeta^2, && (6.56)\\
H_\zeta =& 480.7 + 161.3\,\zeta^2 + 12.6\,\zeta^4 + \mathcal{O}(\zeta^6). && (6.57)
\end{aligned}$$

Proceeding as in the case for $\zeta \gg 1$ we find the constants

$$C_{1,\zeta,\xi} = \frac{\left(-258.6 - 178.8\zeta^2 + 1.3\zeta^4\right)\xi'' + \left(1197.6 + 484.6\zeta^2 + 59.1\zeta^4 + 2.2\zeta^6\right)\chi''}{\left(222.1 - 28.4\zeta^2 + 10.2\zeta^4 - 0.3\zeta^6\right)\xi''\chi''}, \tag{6.58}$$

$$\begin{aligned}
C_{2,\zeta,\xi} =& \frac{\left(5985.1 + 2007.9\zeta^2 + 156.3\zeta^4\right)\zeta^2}{\left(106755 + 22168.6\zeta^2 + 3127.7\zeta^4 + 1157.7\zeta^6 + 82.8\zeta^8 - 3.6\zeta^{10}\right)\chi''}\\
&+ \frac{\left(-17319.2 - 6924.8\zeta^2 - 962.2\zeta^4 - 74.8\zeta^6 - 3.6\zeta^8\right)\zeta^2}{\left(106755 + 22168.6\zeta^2 + 3127.7\zeta^4 + 1157.7\zeta^6 + 82.8\zeta^8 - 3.6\zeta^{10}\right)\xi''}.
\end{aligned} \tag{6.59}$$

These then specify $\psi_R(x)$ in Eq. (6.28). Here we will assume $TL \gg 1$ such that $\chi'' \gg \xi''$. In order to calculate the conductivity in Eq. (6.1) we have to evaluate the integral $\mathcal{I}(\zeta) = \int \mathrm{d}x\, n(x-\zeta)(1-n(x-\zeta))\, \mathfrak{Re}\, \psi_{R,\zeta}(x)/D$. We can expand this expression in orders of ζ, yielding

$$\mathcal{I}(\zeta) \approx 0.01 + 0.0007\, \zeta^2. \tag{6.60}$$

Consequently, the DC conductivity at large temperatures is given by

$$\mathfrak{Re}\, \sigma \approx \frac{-2e}{Eh} T D \frac{\mathcal{I}(\zeta)}{\xi''} = \frac{2e^2}{h} 0.014 \frac{\pi}{T^5} \left(\frac{k_0^2}{U_0}\right)^2. \tag{6.61}$$

If we compare this result with the AC limit in Eq. (5.55), we find that the scattering lengths are almost identical aside from the numerical prefactor which is different by roughly one magnitude i.e. $\tau^{(\mathrm{AC})}_{g5,\zeta\ll 1} \approx \tau^{(\mathrm{DC})}_{g5,\zeta\ll 1}$.

6.3. Inelastic scattering

6.3.1. Derivation of an integral equation for inelastic scattering

We now turn to the inelastic scattering process. Using Eq. (5.4) and the collision integral calculated in the Appendix D, Eq. (10.109), we get

$$\begin{aligned}\psi_{kR} = &- \frac{2\pi g_{1\mathrm{P}}^2 n_{\mathrm{imp}}}{(-i\omega) L^3 f_{kR}(1-f_{kR})} \sum_{k_2,k_{1'},k_{2'}} [2(k_1-k_2)^2 C_{(k_1,R)(k_2,R),(k_{1'},L)(k_{2'},R)} \\ &+ (k_{1'}-k_{2'})^2 \left(C_{(k_1,R)(k_2,L),(k_{1'},R)(k_{2'},R)} + C_{(k_1,R)(k_2,L),(k_{1'},L)(k_{2'},L)}\right)] \\ &- \frac{eE}{(-i\omega)T}.\end{aligned} \tag{6.62}$$

After some formal manipulations we arrive at

$$\psi_R(k) = A_1 \left(B_1(k)\psi_R(k) + \int \mathrm{d}p\, \mathrm{d}q\, M_1(k,p,q)\psi_R(p) \right) - \frac{eE}{(-i\omega)T}, \tag{6.63}$$

where

$$A_1 = - \frac{g_{1\mathrm{P}}^2 n_{\mathrm{imp}}}{2\pi^2(-i\omega)}, \tag{6.64}$$

$$B_1(k) = \int \mathrm{d}p\, \mathrm{d}q\, f^0_{p,R} \frac{1-f^0_{q,L}}{1-f^0_{k,R}} (1-f^0_{k+p+q,R}) \left\{(k-p)^2 + (2q+k+p)^2\right\}, \tag{6.65}$$

$$M_1(k,p,q) = - 4(k+q)(p+q) f^0_{pR} \frac{(1-f^0_{qL})}{(1-f^0_{kR})} (1-f^0_{k+p+q,R}). \tag{6.66}$$

By making the integrals dimensionless and solving for $\psi_{\zeta,R}$ we get

$$\frac{\psi_{\zeta,\xi_1,R}(x)}{D} = \frac{\xi_1}{1-\xi_1 B_{1,\zeta}(x)} \int \mathrm{d}y \mathrm{d}z\, M_{1,\zeta}(x,y,z) \frac{\psi_{\zeta,\xi_1,R}(y)}{D} - \frac{i}{1-\xi_1 B_{1,\zeta}(x)}. \tag{6.67}$$

In this step we defined:

$$\xi_1 = - \frac{n_{\mathrm{imp}}}{2\pi^2(-i\omega)} g_{1\mathrm{P}}^2 T^4, \qquad D = \frac{eE}{\omega T}, \tag{6.68}$$

$$M_{1,\zeta}(x,y,z) = - 4(x+z)(y+z) n(y-\zeta) \frac{1-n(-z-\zeta)}{1-n(x-\zeta)} (1-n(x+y+z-\zeta)), \tag{6.69}$$

$$\begin{aligned}B_{1,\zeta}(x,y,z) = &\left[(x-y)^2 + (2z+x+y)^2\right] n(y-\zeta) \frac{1-n(-z-\zeta)}{1-n(x-\zeta)} \\ &\times (1-n(x+y+z-\zeta)),\end{aligned} \tag{6.70}$$

$$B_{1,\zeta}(x) = \int \mathrm{d}y\, \mathrm{d}z\, B_{1,\zeta}(x,y,z). \tag{6.71}$$

Note that the dimensionless factor ζ would drop out of the expressions completely if we would shift $x \to x+\zeta$, $y \to y+\zeta$ and $z \to z-\zeta$. Therefore, our results are not affected by the ratio of Fermi energy and temperature and ψ will be a function of $(x-\zeta)$ only, i.e. $\psi_{\zeta,\xi_1,R}(x+\zeta) = \psi_{\xi_1,R}(x)$. The calculation of the integrals can be found in Appendix E.2 and yields the approximate expression

$$B_{1,\zeta}(x+\zeta) \equiv B_1(x) = x^4 + \pi^4 \tag{6.72}$$

and for the integral kernel

$$M_1(x,y) = \int \mathrm{d}z\, M_{1,\zeta}(x+\zeta, y+\zeta, z) \approx \sum_{n=0}^{2} x^n \frac{y}{\sinh(y)} a_n(y). \tag{6.73}$$

Here we defined

$$a_0(y) = \frac{2}{3}y^2\,,\;\; a_1(y) = -\frac{8}{3}y \text{ and } \;\; a_2(y) = \frac{2}{3}. \tag{6.74}$$

In the derivation of Eq. (6.73) we made the assumption that x is close to ζ to simplify the kernel.

6.3.2. Solution of the integral equation

Using the results Eq. (6.72) and (6.73) in the integral equation Eq. (6.67) and shifting $x \to x+\zeta$ and $y \to y+\zeta$ we are left with

$$\begin{aligned}\frac{\psi_{\xi_1,R}(x)}{D} =& \frac{\xi_1}{1-\xi_1 B_1(x)} \sum_{n=0}^{2} x^n \int \mathrm{d}y\, \frac{y}{\sinh(y)} a_n(y) \frac{\psi_{\xi_1,R}(y)}{D} \\ &- \frac{i}{1-\xi_1 B_1(x)}.\end{aligned} \tag{6.75}$$

The integral equation, Eq. (6.75), is separable and can therefore be solved exactly. Since we are interested in the DC limit we again only consider the real part of ψ and assume $|\xi_1'' \min_y B(y)| \gg 1$, such that we can approximate the ξ_1 dependance. This assumption specifies the frequency range in which our result is valid to be $\omega^{-1} \gg 0.022\pi^2/(n_{\mathrm{imp}} g_{1\mathrm{P}}^2 T^4)$. This will be the time scale that determines Drude's law in the end.

For now we proceed by mapping our problem to a set of linear equations by defining:

$$\mathfrak{Re}\, \frac{\psi_{\xi_1,R}(x)}{D} \equiv -\frac{1}{B_1(x)} \sum_{n=0}^{2} x^n C^1_{n,\xi_1''} + \frac{1}{\xi_1'' B_1(x)}. \tag{6.76}$$

with[2]

$$C^1_{n,\xi_1''} = E^1_{n,m} C^1_{m,\xi_1''} + F^1_{n,\xi_1''}, \tag{6.77}$$

$$E^1_{n,m} = -\int \mathrm{d}y\, \frac{1}{B_1(y)} \frac{y}{\sinh(y)} a_n(y) y^m, \tag{6.78}$$

$$F^1_{n,\xi_1''} = -\frac{1}{\xi_1''} E^1_{n,0}. \tag{6.79}$$

[2]Summation convention over latin indices is implied.

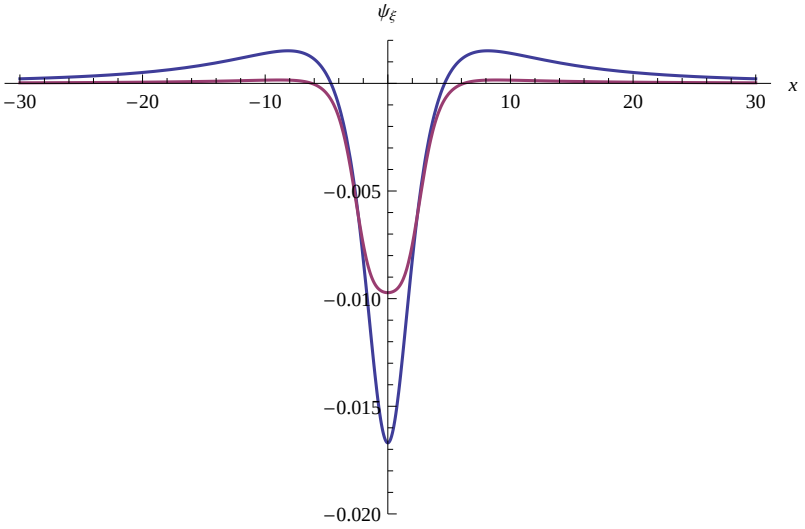

Figure 6.1.: Comparison between the analytical (red curve) and numerical (blue curve) results for the function ψ parametrizing deviations from the equilibrium distribution function due to inelastic scattering processes. For the plot we choose the dimensionless parameter as $\xi = -1$ and x denotes momentum measured in units of temperature.

The integrals specifying E and F can be solved numerically and the resulting system of linear equations is solved for C as in Eq. (6.45). We obtain the result

$$\mathfrak{Re}\,\frac{\psi_{\xi_1,R}(x)}{D} = \frac{1}{\xi_1'' B_1(x)}\left(0.946972 - 0.0245868x^2\right). \tag{6.80}$$

The knowledge of $\psi_R(x)$ now enables us to compute the DC conductivity from Eq. (6.1):

$$\mathfrak{Re}\,\sigma_{\mathrm{DC}} = \frac{2e^2}{h}\tau_{1\mathrm{P}}^{(\mathrm{DC})}, \tag{6.81}$$

where

$$\tau_{1\mathrm{P}}^{(\mathrm{DC})} = \kappa_1 \frac{\pi^2}{n_{\mathrm{imp}} g_{1\mathrm{P}}^2 T^4}. \tag{6.82}$$

Here κ_1 is a numerical factor. Our analytical calculation yields $\kappa_{1,\mathrm{an}} = 1.62 \times 10^{-2}$.

Alternatively, we solve the integral equation in Eq. (6.67) numerically by discretizing the integral and solving the resulting set of linear equations. For the numerical treatment we use the exact forms of the involved functions rather than the approximations used for the analytical analysis. The comparison between both treatments is shown in Fig 6.1. We see that they agree very well, except that the analytical approximation underestimates the amplitude of oscillations. However, it was checked that the dependance of $\psi_{\xi_1''}$ on the dimensionless factor ξ_1'' is captured correctly by the analytical result. Therefore, the only difference to the exact numerical results is a different prefactor. This prefactor can be extracted from the numerical analysis and we find: $\kappa_{1,\mathrm{num}} = 2.3 \times 10^{-2}$.

Lastly, we can compare our results in the DC case with those in the AC case from Sec. 5.6. We find that the scattering times are the same except for a slight discrepancy in the numerical prefactor. However, one has to keep in mind that the analytical treatment is only hand-wavy and does not represent an controllable expansion for $x \approx \zeta$. The numerical treatment on the other hand is exact aside from possible numerical errors but the numerical prefactor is still smaller than in the AC case by roughly a factor of 2.

Having evaluated the high frequency as well as the low frequency asymptotics, we conclude that the conductivity as a function of frequency is marginally non-Drude.

6.4. Two particle process

6.4.1. Derivation of an integral equation for the two particle process

Finally, let us discuss the DC limit of the two particle process. Here we proceed analogously to the single particle case. Using the collision integral from Eq. (10.119) in Eq. (5.4) we get

$$\begin{aligned}\psi_{k_1,R} =& \frac{-2\pi g_{2\mathrm{P}}^2}{(-i\omega) f^0_{k_1,R}(1-f^0_{k_1,R})} \frac{n_{\mathrm{imp}}}{L^3} \sum_{k_2,k_{1'},k_{2'}} (k_{2'}-k_{1'})^2 (k_2-k_1)^2 \\ & \times C_{(k_1,R)(k_2,R),(k_{1'},L)(k_{2'},L)} - \frac{eE}{(-i\omega)T}.\end{aligned} \tag{6.83}$$

After some simplifications we arrive at the equation

$$\psi_{kR} = A_2 B_2(k)\psi_{kR} + A_2 \int \mathrm{d}p\mathrm{d}q \left(M_2(k,p,q) + 2M_2(k,q,-p)\right)\psi_{pR} - \frac{eE}{(-i\omega)T}, \tag{6.84}$$

with

$$A_2 = -\frac{g_{2\mathrm{P}}^2 n_{\mathrm{imp}}}{4\pi^2(-i\omega)}, \tag{6.85}$$

$$M_2(k,p,q) = (2q+k+p)^2(p-k)^2 f^0_{pR}\frac{(1-f^0_{qL})}{(1-f^0_{kR})}(1-f^0_{k+p+q,R}), \tag{6.86}$$

$$B_2(k) = \int \mathrm{d}p\mathrm{d}q\, M_2(k,p,q). \tag{6.87}$$

In dimensionless units this yields

$$\begin{aligned}\frac{\psi_{\zeta,\xi_2,R}(x)}{D} =& \frac{\xi_2}{1-\xi_2 B_{2,\zeta}(x)} \int \mathrm{d}y\, \mathrm{d}z\, \left(M_{2,\zeta}(x,y,z) + 2M_{2,\zeta}(x,z,-y)\right) \frac{\psi_{\zeta,\xi_2,R}(y)}{D} \\ & - \frac{i}{1-\xi_2 B_{2,\zeta}(x)},\end{aligned} \tag{6.88}$$

where the objects are defined as

$$\xi_2 = -\frac{1}{(-i\omega)}\frac{g_{2\mathrm{P}}^2}{4\pi^2} n_{\mathrm{imp}} T^6\,, \tag{6.89}$$

$$\begin{aligned}M_{2,\zeta}(x,y,z) =& (2z+x+y)^2(y-x)^2 n(y-\zeta)\frac{1-n(-z-\zeta)}{1-n(x-\zeta)} \\ & \times (1-n(x+y+z-\zeta)),\end{aligned} \tag{6.90}$$

$$B_{2,\zeta}(x) = \int \mathrm{d}y\mathrm{d}z\, M_{2,\zeta}(x,y,z). \tag{6.91}$$

The various integrals are calculated in the Appendix E.3. We obtain the function B_2,

$$B_{2,\zeta}(x+\zeta) \equiv B_2(x) = \frac{11}{90}x^6 + \frac{\pi^6}{2}, \tag{6.92}$$

and the kernel

$$\begin{aligned}&\int \mathrm{d}z\, \left(M_{2,\zeta}(x+\zeta, y+\zeta, z-\zeta) + 2M_{2,\zeta}(x+\zeta, z-\zeta, -y-\zeta)\right) \\ \approx& \frac{y}{\sinh(y)} \sum_{n=0}^{4} b_n(y) x^n.\end{aligned} \tag{6.93}$$

Here we defined the coefficient functions

$$b_0(y) = \frac{7}{5}y^4\,,\ \ b_1(y) = -\frac{94}{15}y^3\,,\ \ b_2(y) = \frac{176}{15}y^2\,,\ \ b_3(y) = -\frac{94}{15}y\,,\ \ b_4(y) = \frac{7}{5}. \tag{6.94}$$

6.4.2. Solution of the integral equation

First we approximate the expressions in Eq. (6.88) in the DC limit. For $|\xi_2'' \max_y B_{2,\zeta}(y)| >> 1$ we get $\mathfrak{Re}\{\xi_2(1-\xi_2 B_2(y))^{-1}\} \approx -(B_2(y))^{-1}$. This specifies the time scale $\omega^{-1} \gg 8.3\times10^{-3}\pi^2/(n_{\text{imp}}g_{2\text{P}}^2T^6)$ in which our approximations are valid. We proceed by mapping the integral equation Eq. (6.88) to an algebraic equation:

$$\begin{aligned}\mathfrak{Re}\,\frac{\psi_{\xi_2,R}(x)}{D} \approx& -\frac{1}{B_2(x)}\sum_{n=0}^{4}x^n\int \mathrm{d}y\frac{(y)}{\sin(y)}b_n(y)\frac{\psi_{\xi_2,R}(y)}{D} \\ &+\frac{1}{\xi_2''B_2(x)} \end{aligned} \tag{6.95}$$

$$= -\frac{1}{B_2(x)}\sum_{n=0}^{4}x^n C^2_{n,\xi_2''} + \frac{1}{\xi_2''B_2(x)}, \tag{6.96}$$

Here we defined:

$$C^2_{n,\xi_2''} = E^2_{n,m}C^2_{m,\xi_2''} + F^2_{n,\xi_2''}, \tag{6.97}$$

$$E^2_{n,m} = \int \mathrm{d}y\,\frac{1}{B_2(y)}\frac{y}{\sinh(y)}a_n(y)y^m, \tag{6.98}$$

$$F^2_{n,\xi_2''} = -\frac{1}{\xi_2''}E^2_{n,0,\xi_2''}. \tag{6.99}$$

The integrals in E and F can be solved numerically. The resulting system of linear equations is solved for C and the result is

$$\mathfrak{Re}\,\frac{\psi_{\xi_2,R}(x)}{D} = -\frac{1}{\xi_2''B_2(x)}\left(-3.87495 - 0.135023x^2 + 0.0411497x^4\right). \tag{6.100}$$

Inserting this into our general formula for conductivity, Eq. (6.1) and calculating the integral numerically we get

$$\mathfrak{Re}\,\sigma = \kappa_2\frac{2e^2}{h}\frac{1}{\omega\xi_2''} = \frac{2e^2}{h}\tau_{2\text{P}}^{(\text{DC})} \tag{6.101}$$

with the scattering time

$$\tau_{2\text{P}}^{(\text{DC})} = \kappa_2\frac{4\pi^2}{g_{2\text{P}}^2 n_{\text{imp}}T^6}, \tag{6.102}$$

$$\kappa_{2,\text{ana}} = 4.9\times10^{-3}. \tag{6.103}$$

We also solve the integral equation Eq. (6.88) numerically. The comparison with the analytical results is shown in Fig 6.2. While the overall form of ψ is similar in both cases the amplitude of oscillations is overestimated by the analytical solution. However, we checked that the dependance on the dimensionless parameter ξ_2'', that contains all the physical quantities, is captured exactly by the analytical solution and the only difference is in the numerical prefactor, for which we derive a value of $\kappa_{2,\text{num}} = 1.06\times10^{-3}$. Therefore, the scattering time in the DC case Eq. (6.102) matches the one in the AC case Eq. (5.100), except for a small numerical error and we again conclude that the overall conductivity will be slightly non-Drude.

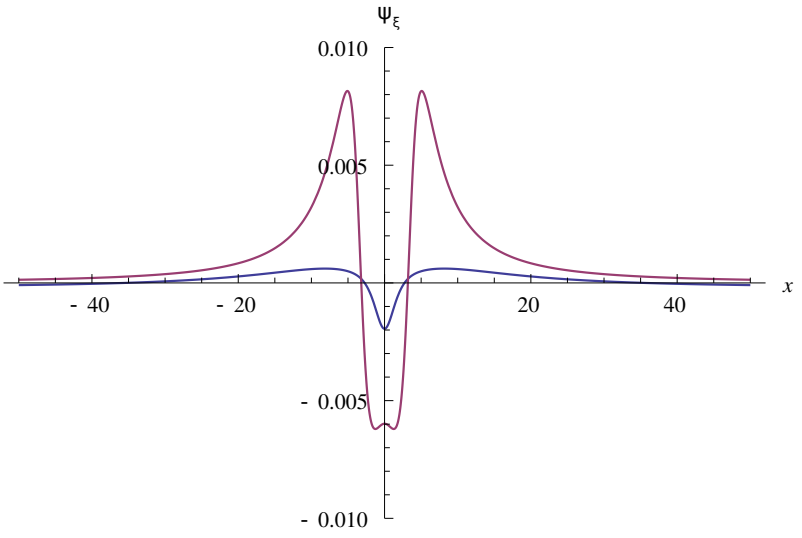

Figure 6.2.: Comparison between the analytical (red curve) and numerical (blue curve) results for the function ψ parametrizing deviations from the equilibrium distribution function in the case of two particle scattering. The parameter ξ is chosen to be $\xi = -1$.

6.5. Summary and results

Let us summarize. We studied the DC limit for g_5, inelastic and two particle processes by deriving exact integral equations from the kinetic equation formalism and solving them approximatively in different regimes of temperature. We find that the DC resistivity is dominated by the contribution due to inelastic scattering processes that yield the corresponding conductivity Eq. (6.81). Furthermore, even in the case of clean systems the DC conductivity is not infinite as opposed to the case of a conventional LL. Indeed, g_5 terms lead to an exponentially big, but finite DC conductivity in the low energy limit, given in Eq. (6.51). Given the low frequency results calculated in this chapter and the high frequency dependance obtained in Ch. 5 we conclude in most cases that the conductivity behaves slightly different than Drude's law, Eq. (5.29). This is because of discrepancies in the numerical prefactors of the scattering time. Moreover the g_5 term shows peculiar behavior at low temperatures due to the special state at zero momentum. This is discussed in Sec. 6.2.2. The results and the comparison with the AC limit are summarized in Tab. 6.1.

process	τ in the AC limit		τ in the DC limit	
	$T \ll k_F$	$T \gg k_F$	$T \ll k_F$	$T \gg k_F$
g_5	$\frac{45\pi}{22}\left(\frac{k_0^2}{V_0 k_F^3}\right)^2 T e^{k_F/T}$	$6.5\times10^{-3}\left(\frac{k_0^2}{V_0}\right)^2 T^{-5}$	$0.254\frac{\pi}{k_F^5}\left(\frac{k_0^2}{U_0}\right)^2 e^{\frac{k_F}{T}}$	$0.014\pi\left(\frac{k_0^2}{U_0}\right)^2 T^{-5}$
1P	$9.62\times10^{-3}\frac{\pi^2}{n_{\text{imp}}}\frac{1}{g_{1\text{P}}^2}T^{-4}$		$2.3\times10^{-2}\frac{\pi^2}{n_{\text{imp}}}\frac{1}{g_{1\text{P}}^2}T^{-4}$	
2P	$5.69\times10^{-4}\frac{\pi^2}{n_{\text{imp}}}\frac{1}{g_{2\text{P}}^2}T^{-6}$		$4.23\times10^{-3}\frac{\pi^2}{n_{\text{imp}}}\frac{1}{g_{2\text{P}}^2}T^{-6}$	

Table 6.1.: Comparison of the most relevant scattering mechanisms and their respective scattering time τ in different regimes of temperature and frequency. In the AC limit the correction to conductivity can then be obtained as: $\sigma^{(1)} = -\frac{2e^2}{(-i\omega)^2 h}\tau^{-1}$. On the other hand, in the DC limit the conductivity obeys Drude's law: $\sigma = \frac{2e^2}{h}\tau$. The g_5 processes are purely due to electron-electron interaction while inelastic 1P and two-particle processes are effective Hamiltonians describing combined effects of disorder and interaction. The coupling constants for inelastic and two-particle scattering are $g_{1\text{P}} = \frac{2k_F V_0 U_0}{\pi k_0^2}$ and $g_{2\text{P}} = \frac{8k_F^2 U_0 V_0}{\pi k_0^4}$.

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7. Luttinger liquid effects

So far we have discussed transport properties of one-dimensional electrons subject to weak interaction and impurity scattering neglecting LL effects due to the presence of a finite g_2 term. We find that the temperature dependance of the conductivity behaves as a power law with integer powers. However, in a generic one-dimensional LL one would expect power laws where the exponent depends on the Luttinger parameter K. In this chapter we include LL effects in our previous treatment by bosonizing the model and obtaining the scaling dimensions of the different operators in Sec. 7.1. These are then used to calculate the renormalized coupling constants of the effective model containing only inelastic and two particle processes in Sec. 7.2.

7.1. Bosonization of the effective Hamiltonian

We start by bosonizing the effective Hamiltonians derived in Sec. 5.5:

$$\hat{H}_{1\mathrm{P}}(x=0) = g_{1\mathrm{P}} \left[\left(-i\partial_x \psi_L^\dagger \psi_L + i\psi_R^\dagger \partial_x \psi_R \right) \psi_L^\dagger \psi_R + h.c \right], \tag{7.1}$$

$$\hat{H}_{2\mathrm{P}}(x=0) = g_{2\mathrm{P}} \left[\psi_L^\dagger \partial_x \psi_L^\dagger \psi_R \partial_x \psi_R + h.c \right]. \tag{7.2}$$

To bosonize the operators we exploit the fact that they are local. For instance we can rewrite the first part of $\hat{H}_{1\mathrm{P}}$ as

$$-i\partial_x \psi_L^\dagger \psi_L \psi_L^\dagger \psi_R = -i\partial_x \left(\psi_L^\dagger \psi_L \right) \psi_L^\dagger \psi_R \tag{7.3}$$

$$= -i \left(\partial_x J_L \right) \psi_L^\dagger \psi_R \tag{7.4}$$

$$= \frac{i}{4\pi^{3/2}} \partial_x^2 \phi_L e^{-i\sqrt{4\pi}(\phi_L + \phi_R)}. \tag{7.5}$$

In the first step we used the fact that a local operator like $\psi_L^\dagger(\partial_x \psi_L)\psi_L^\dagger \psi_R$ vanishes because of the fermionic anticommutation relations. Next we expressed $\psi_L^\dagger \psi_L$ as the left moving current J_L whose bosonization relation is known from Eq. (3.51) and we used the bosonization relations Eq. (3.49) to translate the other operators to the bosonic language. For the second operator in $\hat{H}_{1\mathrm{P}}$ we can use analogous reasoning to get

$$i\psi_R^\dagger \partial_x \psi_R \psi_L^\dagger \psi_R = i\partial_x \left(\psi_R^\dagger \psi_R \right) \psi_L^\dagger \psi_R \tag{7.6}$$

$$= -\frac{i}{4\pi^{3/2}} \partial_x^2 \phi_R e^{-i\sqrt{4\pi}(\phi_L + \phi_R)}. \tag{7.7}$$

If we combine Eq. (7.5) and Eq. (7.7) we obtain the bosonized form of the inelastic scattering term:

$$\hat{H}_{1\mathrm{P}} = \frac{g_{1\mathrm{P}}}{4\pi^{3/2}} \left[-i\partial_x^2 \left(\phi_R - \phi_L\right) e^{-i\sqrt{4\pi}(\phi_L+\phi_R)} + h.c. \right] \tag{7.8}$$

$$= \frac{g_{1\mathrm{P}}}{4\pi^{3/2}} \left[-i\partial_x^2 \theta e^{-i\sqrt{4\pi}\varphi} + h.c. \right]. \tag{7.9}$$

For the two particle scattering term we have to treat operators like $(\psi_L\psi_L)(x=0)$. Let us define $\bar{\epsilon} = \bar{z}' - \bar{z}$ and the corresponding relations for the derivatives, $\partial_{\bar{z}} = -\partial_{\bar{\epsilon}}$ and $\partial_{\bar{z}'} = \partial_{\bar{\epsilon}}$, where the imaginary variable z was introduced in Eq. (3.3). With this notation the product of vertex operators defined in Eq. (3.28) yields

$$\psi_L(\bar{z})\psi_L(\bar{z}') = \frac{1}{2\pi} e^{-i\sqrt{4\pi}\phi_L(\bar{z})} e^{-i\sqrt{4\pi}\phi_L(\bar{z}')} \tag{7.10}$$

$$= \frac{\bar{\epsilon}}{2\pi} e^{-i\sqrt{4\pi}(\phi_L(\bar{z})+\phi_L(\bar{z}'))}. \tag{7.11}$$

Consequently, the following relations hold:

$$\partial_{\bar{z}}\psi_L(\bar{z})\psi_L(\bar{z}') = (-1 - i\bar{\epsilon}\sqrt{4\pi}\partial_{\bar{z}}\phi_L(\bar{z})) \frac{1}{2\pi} e^{-i\sqrt{4\pi}(\phi_L(\bar{z})+\phi_L(\bar{z}'))}, \tag{7.12}$$

$$\partial_{\bar{z}'}\psi_L^\dagger(\bar{z})\psi_L^\dagger(\bar{z}') = (1 - i\bar{\epsilon}\sqrt{4\pi}\partial_{\bar{z}'}\phi_L(\bar{z}')) \frac{1}{2\pi} e^{-i\sqrt{4\pi}(\phi_L(\bar{z})+\phi_L(\bar{z}'))}. \tag{7.13}$$

Therefore, we can calculate the local operator $\psi_L\psi_L$ by employing the trick

$$2(\partial_{\bar{z}}\psi_L)\psi_L = \lim_{\bar{z}\to\bar{z}'} \left[\partial_{\bar{z}}\psi_L(\bar{z})\psi_L(\bar{z}') - \psi_L(\bar{z})\partial_{\bar{z}'}\psi_L(\bar{z}') \right] \tag{7.14}$$

$$= -\frac{1}{\pi} e^{-i4\sqrt{\pi}\phi_L}. \tag{7.15}$$

Back in space-time coordinates we obtain

$$\partial_x\psi_L\psi_L = \frac{i}{\pi} e^{-i4\sqrt{\pi}\phi_L}. \tag{7.16}$$

For the corresponding term with right movers we proceed analogously to derive

$$2(\partial\psi_R)\psi_R = i(\partial_x\psi_R)\psi_R = \lim_{z\to z'} \left[\partial\psi_R(z)\psi_R(z')) - \psi_R(z)\partial\psi_R(z')) \right] \tag{7.17}$$

$$= -\frac{1}{\pi} e^{-i\sqrt{4\pi}2\phi_R}, \tag{7.18}$$

which yields the result

$$\partial_x\psi_R\psi_R = -\frac{i}{\pi} e^{-i\sqrt{4\pi}2\phi_R}. \tag{7.19}$$

Summarizing, the bosonized form of $\hat{H}_{2\mathrm{P}}$ is

$$\hat{H}_{2\mathrm{P}} = g_{2\mathrm{P}} \left[\psi_L^\dagger \partial_x \psi_L^\dagger \psi_R \partial_x \psi_R + h.c \right] \tag{7.20}$$

$$= \frac{1}{\pi^2} g_{2\mathrm{P}} \left[e^{-i2\sqrt{4\pi}(\phi_R+\phi_L)} + h.c. \right] \tag{7.21}$$

$$= \frac{1}{\pi^2} g_{2\mathrm{P}} \left[e^{-i2\sqrt{4\pi}\varphi} + h.c. \right]. \tag{7.22}$$

Process	Fermionic	Bosonized	Δ
Inelastic	$g_{1\mathrm{P}}\left[\left(-i\partial_x\psi_L^\dagger\psi_L + i\psi_R^\dagger\partial_x\psi_R\right)\psi_L^\dagger\psi_R + h.c\right]$	$\frac{g_{1\mathrm{P}}}{4\pi^{3/2}}\left[-i\partial_x^2\theta e^{-i\sqrt{4\pi}\varphi} + h.c.\right]$	K+2
Two particle	$g_{2\mathrm{P}}\left[\psi_L^\dagger\partial_x\psi_L^\dagger\psi_R\partial_x\psi_R + h.c\right]$	$\frac{1}{\pi^2}g_{2\mathrm{P}}\left[e^{-i2\sqrt{4\pi}\varphi} + h.c.\right]$	4K

Table 7.1.: Scattering terms and their scaling dimensions.

7.2. Analysis of the scaling behavior

The complete model now consists of $\hat{H} = \hat{H}_0 + \hat{H}_2 + \hat{H}_{1\mathrm{P}} + \hat{H}_{2\mathrm{P}}$. As we argued in Sec. 3.5.1 the free Hamiltonian $\hat{H}_0$ can be brought back into canonical form after bosonization by scaling the fields as $\varphi \to \sqrt{K}\varphi$ and $\theta \to 1/\sqrt{K}\theta$. We use the rescaled operators, the scaling dimensions of vertex operators Eq. (3.31) and the fact that a derivative raises the scaling dimension by one to obtain the scaling dimension of $\hat{H}_{1\mathrm{P}}$ and $\hat{H}_{2\mathrm{P}}$ from the Eqs. (7.7) and (7.22):

$$\Delta_{1\mathrm{P}} = K + 2, \quad \Delta_{2\mathrm{P}} = 4K. \tag{7.23}$$

The different operators and their scaling dimensions are summarized in Tab. 7.1. In Sec. 3.1.3 we showed that the coupling constant of a conformal operator scales under the RG as

$$g'(b) = gb^{1-\Delta}. \tag{7.24}$$

Note that we are dealing with local operators, which do not contain any spatial integration. Consequently, Eq. (7.24) differs from Eq. (3.15) by a factor of b. Here the factor b denotes the ratio of momentum space cutoffs before and after the RG procedure: $b = \Lambda/\Lambda_{\mathrm{eff}}$. In our model of helical fermions, the original momentum cutoff is given by $\Lambda = k_0$.[1] The RG has to be stopped at an effective cutoff Λ_{eff} given by the largest energy scale in the problem.

In our case we have a competition between the temperature T, the inverse transport scattering time τ^{-1} and the frequency ω. Because we assume weak disorder, the condition $T\tau \gg 1$ always holds. Consequently, we can have the following two hierarchies of energy scales in the AC limit: (i) $T \gg \omega \gg \tau^{-1}$ and (ii) $\omega \gg T \gg \tau^{-1}$. Furthermore, in the DC limit the frequency vanishes and we are left with the condition (iii) $T \gg \tau^{-1} \gg \omega$.

In the cases (i) and (iii) the effective cutoff is given by the temperature, $\Lambda_{\mathrm{eff}} = T$ and we obtain $b = k_0 T^{-1}$. Therefore, the RG equation Eq. (7.24) becomes

$$g(T) = g(0)\left(Tk_0^{-1}\right)^{\Delta-1}. \tag{7.25}$$

On the other hand, in the regime (ii) the RG is stopped by frequency and we have

$$g(\omega) = g(0)\left(\omega k_0^{-1}\right)^{\Delta-1}. \tag{7.26}$$

Using these RG equation and the scaling dimensions Eq. (7.23), we can include Luttinger liquid effects into our previous calculations through the renormalized coupling constants. We have to be careful however for there is one further sublety. The structure of the kinetic equation already incorporates the engineering dimension of the respective operator. For example the scattering time due to the inelastic processes is proportional to $(T/k_0)^{-4} = (T/k_0)^{-2(\Delta_{1\mathrm{P},K=1}-1)}$.

[1] This is true for a generic HLL. If the HLL describes the edge states of a 2D TI, the cutoff of the theory is given by the bulk gap.

Consequently, we have to split the scaling dimension of an operator into the engineering dimension (given by the value at K=1) and the rest, the anomalous dimension. We obtain

$$\Delta_{1\mathrm{P,e}} = 3, \quad \Delta_{1\mathrm{P,an}} = K - 1, \tag{7.27}$$

$$\Delta_{2\mathrm{P,e}} = 4, \quad \Delta_{2\mathrm{P,an}} = 4K - 4. \tag{7.28}$$

While the engineering dimension always scales with temperature, the anomalous dimension scales with the highest energy scale of the problem. In the regimes (i) and (iii) the highest energy scale is given by the temperature while in the regime (ii) we have $\Lambda_{\mathrm{eff}} = \omega$. The kinetic equation already contains the part of the coupling constant that scales as $\left(Tk_0^{-1}\right)^{\Delta_\mathrm{e}-1}$. Therefore Luttinger liquid effects arise only from the anomalous part that scales as $\left(\Lambda_{\mathrm{eff}}k_0^{-1}\right)^{\Delta_{\mathrm{an}}}$. Because of the fact that the inverse scattering time depends on the square of the coupling constant, i.e. $\tau \sim g^{-2}$, we obtain altered power laws for the temperature dependance due to the anomalous scaling dimensions

$$\tau_{1\mathrm{P}} \sim \left(\frac{\Lambda_{\mathrm{eff}}}{k_0}\right)^{-2K+2}, \quad \tau_{2\mathrm{P}} \sim \left(\frac{\Lambda_{\mathrm{eff}}}{k_0}\right)^{-8K+8}. \tag{7.29}$$

Let us summarize. If we include the renormalized coupling constants, the expressions for the scattering time listed in Tab. 6.1 gain non-universal power laws, where the exponent depends on the Luttinger liquid constant K. This in turn affects the conductivity in the AC and DC limit. For completeness we state the explicit form of the conductivity in the considered regimes:

- (i) AC limit, where $T \gg \omega \gg \tau^{-1}$

$$\sigma_{1\mathrm{P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 103.902\frac{32}{\pi^4}\frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} k_F^2 (V_0U_0)^2 \left(\frac{T}{k_0}\right)^{2K+2}, \tag{7.30}$$

$$\sigma_{2\mathrm{P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 1757.97\frac{64}{\pi^4} n_{\mathrm{imp}} \frac{2e^2}{h(-i\omega)^2} k_F^4 \left(U_0V_0\right)^2 \left(\frac{T}{k_0}\right)^{8K-2}. \tag{7.31}$$

- (ii) AC limit, where $\omega \gg T \gg \tau^{-1}$

$$\sigma_{1\mathrm{P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 103.902\frac{32}{\pi^4}\frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} k_F^2 (V_0U_0)^2 \left(\frac{T}{k_0}\right)^4 \left(\frac{\omega}{k_0}\right)^{2K-2}, \tag{7.32}$$

$$\sigma_{2\mathrm{P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 1757.97\frac{64}{\pi^4} n_{\mathrm{imp}} \frac{2e^2}{h(-i\omega)^2} k_F^4 \left(U_0V_0\right)^2 \left(\frac{T}{k_0}\right)^6 \left(\frac{\omega}{k_0}\right)^{8K-8}. \tag{7.33}$$

- (iii) DC limit, $T \gg \tau^{-1} \gg \omega$

$$\sigma_{1\mathrm{P}}^{(\mathrm{DC})} = 2.3 \times 10^{-2} \frac{\pi^4}{n_{\mathrm{imp}}} \frac{1}{4k_F^2 U_0^2 V_0^2} \left(\frac{T}{k_0}\right)^{-2K-2}, \tag{7.34}$$

$$\sigma_{2\mathrm{P}}^{(\mathrm{DC})} = 4.23 \times 10^{-3} \frac{\pi^4}{n_{\mathrm{imp}}} \frac{1}{64k_F^4 U_0^2 V_0^2} \left(\frac{T}{k_0}\right)^{-8K+2}. \tag{7.35}$$

While the unusual frequency dependance in regime (ii) is most interesting a physical interpretation of the result was not within the scope of this thesis.

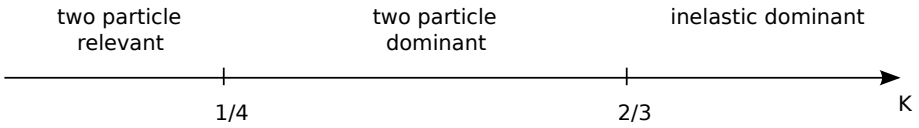

Figure 7.1.: Parameter ranges of the LL parameter in which certain scattering mechanism are dominant.

Apart from the renormalization of the power laws determining conductivity we can also use the scaling dimensions Eq. (7.23) to make general statements about the respective relevance of the two operators. Applying our results from Sec. 3.1.3 to the case of local operators we can argue that an operator with scaling dimension Δ is relevant if $\Delta < 1$ and irrelevant if $\Delta > 1$. Consequently, the single particle process is always irrelevant while the two particle process is irrelevant for $K > 1/4$ and becomes relevant for $K < 1/4$. For the parameter regime $K > 2/3$ the inelastic scattering process provides the primary scattering mechanism. However, for $K < 2/3$ the two particle process becomes dominant. This is in agreement with our treatment of weakly interacting electrons in Sec. 5.6. There we find that the inelastic process leads to conductivity corrections scaling as T^4 in the low temperature limit. They are therefore more relevant than the corresponding corrections due to two particle scattering that go as T^6. The parameter ranges of the LL parameter K for the importance of different scattering mechanism are depicted in Fig. 7.1.

8. Summary and Conclusion

The goal of this work was to study transport in the edge channels of two dimensional topological insulators under the combined effect of disorder and interactions. To accomplish this, we first derived a microscopic model describing one-dimensional helical fermions in Sec. 2.2.4. There we considered one-dimensional electrons subject to interactions in the density-density channel and a local, nonmagnetic impurity. Furthermore, due to the presence of spin-orbit coupling the S_z symmetry of the electrons is broken. To characterize transport properties we calculated the conductivity of helical fermions in the high frequency and low frequency case. This was done using a kinetic equation approach in an infinite wire geometry and for different regimes of temperature. Finally, including Luttinger liquid effects we find altered power laws of the temperature dependance of conductivity.

AC conductivity

In Ch. 5 we calculated the AC conductivity of an infinite wire of helical fermions. At first we neglected Luttinger liquid effects which enabled us to use a kinetic equation to describe electronic quasiparticles. Scattering terms were included using a generalized Fermi's golden rule. We did perturbation theory in (i) the T-matrix, valid for weak disorder and interaction strength, and (ii) high frequencies.

Without any scattering mechanism the conductivity is given by: $\sigma^{(0)} = 2e^2/h(-i\omega)$, i.e. the conductance is perfectly quantized. In a clean system there are corrections to this result due to g_5 interaction processes that scale as T^5 at high temperatures (Eq. 5.54) and are exponentially suppressed, as can be shown using phase space arguments, at low temperatures (Eq. 5.51). As expected impurity scattering alone does not influence conductivity because of the topological protection. In the second order of the T-matrix the leading contribution in the low temperature regime originates from combined effects of disorder and interactions and gives rise to corrections scaling as T^4. Our results can be mapped to a short wire, with length L_{eff} by the formal substitution $(-i\omega) \to L_{\text{eff}}^{-1}$. This enables us to make predictions about the conductance of such systems as well. We find that the results of our perturbation theory are in agreement with Ref. [11].

Complementary, we derived an effective Hamiltonian incorporating combined effects of disorder and impurity scattering using an operator product expansion. The obtained terms describe the most relevant scattering mechanism in the low energy regime and agree with the terms considered in Ref. [12]. Corrections to conductivity were derived using the same kinetic equation formalism as before and we find that the leading order correction

comes from inelastic scattering processes and goes as T^4:

$$\sigma_{\mathrm{AC}}^{(1)} = -103.902 \frac{32}{\pi^4} \frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} \left(\frac{k_F V_0 U_0}{k_0^2} \right)^2 T^4. \tag{8.1}$$

Although this approach yields the same power law in temperature as the perturbative one, we find corrections that are lower order in impurity strength and momentum cutoff k_0 compared to the perturbation theory. This is not yet understood and has to be studied in future work.

DC conductivity

Ch. 6 was devoted to the study of transport in the low frequency limit. We studied the DC limit for g_5, inelastic and two particle processes by deriving exact integral equations from the kinetic equation formalism and solving them approximatively in different regimes of temperature. The combination of this analytical treatment and numerics allows us to calculate the exact DC conductivity for the most relevant cases. We find that the resistivity is dominated by the contribution due to inelastic scattering processes. The corresponding DC conductivity is given by:

$$\sigma_{\mathrm{DC}} = \frac{2e^2}{h} 2.3 \times 10^{-2} \frac{\pi^4}{4 k_F^2 n_{\mathrm{imp}}} \left(\frac{k_0^2}{U_0 V_0} \right)^2 T^{-4}. \tag{8.2}$$

Furthermore, even in the case of clean systems the DC conductivity is not infinite as opposed to the case of a conventional Luttinger liquid. Indeed, g_5 terms lead to an exponentially big, but finite DC conductivity in the low temperature limit, given in Eq. (6.51).

Finally, comparing the low frequency results and the high frequency dependance we find slight discrepancies in the numerical prefactors of the scattering time. Therefore we conclude that in most cases the conductivity behaves marginally different than expected from Drude's law, Eq. (5.29). Moreover, the g_5 term shows peculiar behavior at low temperatures due to the special state at zero momentum. This is discussed in Sec. 6.2.2. All results and the comparison between the AC and DC limit are depicted in Tab. 6.1.

Luttinger liquid effects

Lastly, in Ch. 7.1 we bosonized the effective model and used scaling arguments to show that Luttinger liquid effects will renormalize the power laws governing the temperature dependance of conductivity. For weak impurity strength, $T\tau \gg 1$ we discuss three different hierarchies of energy scales and the respective results for conductivity. For the most relevant case we find:

- (i) AC limit, where $T \gg \omega \gg \tau^{-1}$

$$\sigma_{\mathrm{1P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 103.902 \frac{32}{\pi^4} \frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} k_F^2 (V_0 U_0)^2 \left(\frac{T}{k_0} \right)^{2K+2}. \tag{8.3}$$

- (ii) AC limit, where $\omega \gg T \gg \tau^{-1}$

$$\sigma_{\mathrm{1P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 103.902 \frac{32}{\pi^4} \frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} k_F^2 (V_0 U_0)^2 \left(\frac{T}{k_0} \right)^4 \left(\frac{\omega}{k_0} \right)^{2K-2}. \tag{8.4}$$

- (iii) DC limit, $T \gg \tau^{-1} \gg \omega$

$$\sigma_{\mathrm{1P}}^{(\mathrm{DC})} = 2.3 \times 10^{-2} \frac{\pi^4}{n_{\mathrm{imp}}} \frac{1}{4 k_F^2 U_0^2 V_0^2} \left(\frac{T}{k_0} \right)^{-2K-2}. \tag{8.5}$$

Here K is the Luttinger liquid parameter. Additionally, we classify parameter regimes of K where scattering processes may become relevant.

Outlook

The current experimental situation (see Ref. [6]) suggests that the conductance of a long wire is smaller than that of a short wire by roughly one magnitude. Furthermore, there is no evidence of a temperature dependant conductance. This is in clear contrast to the developed theory. All scattering mechanisms clearly give temperature dependant results for transport properties. The one exception to this is the g_5 interaction which was shown to be temperature independant in some regime of system length (see Sec. 6.2). This regime has to be studied in more detail. In particular it might become important to include boundary conditions allowing for a large but finite system length. Another possibility is that weak localisation corrections lead to the increase in resistance.

The theory developed in Refs. [43, 44, 45] gives a fairly complete overall picture of localization effects in usual one-dimensional systems. At high temperatures, interaction-induced dephasing is strong, and one may consider the limit when the dephasing time τ_ϕ is much less than the transport scattering time $\tau_{\rm tr}$. In this limit one nevertheless finds a negative weak-localization correction to conductance due to constructive interference of coherent electron paths. In order to be coherent, these paths originate from impurity configurations which are anomolously close together - the low probability of such configurations being reflected in the parametrically small prefactor $(\tau_\phi/\tau_{\rm tr})^2$ in front of the weak localization correction. The formalism for treating this limit is based on a perturbative treatment of disorder within a functional bosonization approach – this method is very general and has been recently extended to two-leg ladders [46].

The difference between this case and the HLL is that there is no elastic scattering off disorder in the HLL - which means that at least phenomenologically, the disorder scattering itself contributes to the dephasing time. The insight we have gained from our kinetic equation approach may be used as input to adapt the functional bosonization approach, where we fully expect that the weak-localization correction will still be present, but with a modified prefactor. Going beyond this weak-localization regime, the question then remains: can the limit of $\tau_\phi/\tau_{\rm tr} \sim 1$ ever be reached in such a system? In the usual model of quantum wires, the two parameters could be controlled independently: $\tau_{\rm tr}$ is a function of disorder strength, while τ_ϕ depends on the strength of interactions; and both also depend on temperature. On the other hand, in the present case of no elastic scattering and only inelastic scattering off the impurities, then $\tau_{\rm tr}$ acquires a crucial dependence on the strength of interactions, while τ_ϕ presumably is affected by disorder. When these dependencies are better understood, one would be in a position to answer the question on whether a strong localization regime (which occurs when $\tau_\phi/\tau_{\rm tr} \gtrsim 1$) may ever be reached in this system. In other words, one would be able to explore whether the topological protection from localization necessarily persists in the presence of interaction, or else, it may be broken.

All in all, the study of topological insulators remains an exciting research field of contemporary condensed matter theory, which continues to pose many interesting challenges to theorists.

9. Deutsche Zusammenfassung

9.1. Motivation

Topologie, Unordnung und starke Korrelationen in niedrigdimensionalen Systemen sind paradigmatische Konzepte in der modernen Theorie kondensierter Materie. Materialien in welchen die Kombination dieser Elemente eine herausragende Rolle spielt, sind zweidimensionale $\mathbb{Z}_2$ topologische Isolatoren [1, 2, 3, 4, 5]. $\mathbb{Z}_2$ topologische Isolatoren sind neuartige Zustände von Quantenmaterie, die nicht kontinuierlich mit gewöhnlichen Isolatoren und Halbleitern in Verbindung gebracht werden können. Charakteristisch für diese Klasse von Materialien ist eine isolierende Bandlücke im Kristallinneren und leitende Rand- oder Oberflächenzustände. Diese Randzustände unterscheiden sich fundamental von bekannten eindimensionalen Quantenflüssigkeiten, die durch die Theorie der Luttingerflüssigkeit beschrieben werden. Durch die Kombination von Zeitumkehrinvarianz und starker Spin-Bahn-Kopplung sind die Randzustände topologisch vor Streuung durch Unordnung geschützt und besitzen überdies die ungewöhnliche Eigenschaft, dass Spin und Impuls der Quasiteilchen der Flüssigkeit gekoppelt sind. Aufgrund dieser wohldefinierten Projektion des Spins auf die Impulsrichtung bezeichnet man die eindimensionalen Randzustände als „helikale Luttinger Flüssigkeit“.

In solchen Systemen (zuerst in HgTe/CdTe Heterostrukturen experimentell nachgewiesen [5]) wurden zwei unterschiedliche isolierende Phasen gefunden, die beide eine isolierende Bandlücke im Kristallinneren besitzen, sich aber in den Eigenschaften der Randzustände unterscheiden. Während die gewöhnliche, isolierende Phase keine Randzustände besitzt, ist der topologisch nichttriviale Isolator durch die Existenz von paarweise zeitumgekehrten, delokalisierten Randzuständen charakterisiert, welche die Bandlücke überbrücken. Eine solche Konfiguration weist den Quanten-Spin-Hall-Effekt (QSHE) auf, der theoretisch für ein Modellsystem von Graphen mit Spin-Bahn-Kopplung vorhergesagt wurde [3]. Der Phasenübergang zwischen den beiden Isolatorphasen geschieht durch Invertieren der Bandlücke [4].

Transporteigenschaften von HgTe/CdTe Strukturen in der topologisch nichttrivialen Phase wurden in Ref. [5] untersucht. Die Autoren fanden, dass das System einen Phasenübergang zum topologischen Isolator vollzieht, sobald die Breite der HgTe Schicht einen kritischen Wert überschreitet. In diesem Regime wurde ein Leitwert von nahezu dem Leitwertquant $2e^2/h$ gemessen, was für ballistischen Transport in QSH Zuständen zu erwarten ist. Jedoch wurden auch merkliche Abweichungen von diesem Wert für zunehmende Systemgrößen festgestellt. So berichten aktuelle Experimente [6] von Transportmessungen über sehr

lange Distanzen (von der Ordnung 1mm) bei denen der Leitwert um eine Größenordnung von dem zu erwarteten Leitwertquant abweicht. Mögliche Ursachen für dieses Verhalten, Zweiteilchen-Streuprozesse oder Kondo-Verunreinigungen, wurden in Ref. [7, 8, 9, 10] diskutiert. Zudem wurden kürzlich die Auswirkungen von inelastischen Streuprozessen auf das Transportverhalten untersucht [11, 12]. Alle Veröffentlichungen beschränken sich jedoch auf die Erforschung kleiner Systemlängen, wohingegen makroskopische Proben noch nicht theoretisch untersucht wurden.

Neben zweidimensionalen topologischen Isolatoren wurde auch die Existenz dreidimensionaler $\mathbb{Z}_2$ topologischer Isolatoren experimentell nachgewiesen, z.B. bei der Untersuchung von $Bi_{1-x}Sb_x$ Kristallen in Ref. [13]. In diesem Fall führt die Oberfläche zur Existenz eines 2D topologisch geschützten Metalls. Ähnlich zum 2D Fall führt die Inversion der 3D Bandlücke zur Entstehung einer ungeraden Anzahl zweidimensionaler Oberflächenmoden [14]. Zahlreiche experimentelle Gruppen untersuchen zur Zeit die Transporteigenschaften dieser Oberflächenzustände.

Auch Supraleiter können topologischen Charakter besitzen. In diesem Fall hat der Stoff eine supraleitende Bandlücke im Kristallinneren, besitzt aber leitende Quasiteilchen an der Oberfläche. Betrachtet man Spin-Triplet-Supraleiter so haben diese Quasiteilchen den Charakter von Majorana-Fermionen [15]. Da derartige Systeme zum Aufbau topologischer Quantenrechner dienen könnten, erhalten sie im Moment große Aufmerksamkeit.

In dieser Diplomarbeit untersuchen wir den kombinierten Einfluss von Störstellen und Wechselwirkung auf makroskopische Proben zweidimensionaler, topologischer Isolatoren. Abgesehen von der Bedeutung für die Grundlagenforschung ist die systematische Untersuchung dieser Effekte auch von großem Interesse für potenzielle technologische Anwendungen wie z.B. im Bereich der Spintronik. Um die Effekte von Unordnung und Wechselwirkung zu beschreiben stellen wir zunächst ein mikroskopisches Modell für helikale Fermionen unter dem Einfluss von schwacher Unordnung und Wechselwirkung sowie mit gebrochener S_z Symmetrie auf. Daraufhin berechnen wir die Gleichstrom- und Wechselstromleitfähigkeit in verschiedenen Temperaturbereichen durch Lösung der quasiklassischen Boltzmann-Gleichung. Dies ermöglicht es uns Aussagen über die Robustheit des topologischen Schutzes zu treffen und die relevantesten Streumechanismen zu identifizieren. Schließlich binden wir Luttinger-Flüssigkeitseffekte mit ein, die das algebraische Verhalten der Temperaturabhängigkeit der Leitfähigkeit verändern.

9.2. Aufbau der Diplomarbeit

Diese Diplomarbeit ist wie folgt strukturiert. Im ersten Teil, Kap. 2-4, werden die Grundlagen für die weitere Arbeit geschaffen.

Kap. 2 beschäftigt sich mit topologischen Isolatoren. Zunächst geben wir einen Überblick über die Rolle der Topologie in der Theorie kondensierter Materie. Anschließend besprechen wir $\mathbb{Z}_2$ topologische Isolatoren und ihre experimentelle Realisierung. Insbesondere zeigen wir, dass die leitenden Zustände, die an den Rändern des Isolators entstehen, durch eindimensionale helikale Fermionen beschrieben werden können.

Dies stellt die Motivation dar um uns in Kap. 3 mit den Grundlagen eindimensionaler Quantenphysik vertraut zu machen. In diesem Kapitel besprechen wir die Effekte von Wechselwirkung und Unordnung in niedrigdimensionalen Systemen. Zudem werden wichtige Modelle und theoretische Werkzeuge zur Beschreibung eindimensionaler Physik diskutiert.

Schließlich stellen wir in Kap. 4 den Keldysh Formalismus vor, der in der Lage ist Systeme im Nicht-Gleichgewicht zu beschreiben. Zunächst leiten wir die Boltzmann-Gleichung aus

der übergeordneten Theorie her um sie dann auf helikale Fermionen anzuwenden. Die Ergebnisse bilden die Grundlage für die Berechnung von Transporteigenschaften helikaler Fermionen, die im Hauptteil durchgeführt wird.

Das Ziel dieser Diplomarbeit ist die Untersuchung von Transporteigenschaften helikaler Randzustände unter dem Einfluss von Unordnung und Wechselwirkung. Diese Zielsetzung wird in drei Schritten erreicht. Zunächst berechnen wir die Leitfähigkeit für hohe Frequenzen durch Lösen der quasiklassischen Boltzmanngleichung in Kap. 5. Zudem nutzen wir Methoden der konformen Feldtheorie um einen effektiven Hamiltonoperator herzuleiten, der die wichtigsten Streumechanismen beinhaltet. Dies erlaubt uns die Ergebnisse beider Ansätze zu vergleichen. In Kap. 6 nutzen wir sowohl exakte analytische Umformungen als auch numerische Berechnungen um die Integro-Differenzialgleichungen zu lösen, die aus der Boltzmann-Gleichung herleitet werden. Dies erlaubt es uns Aussagen über die Leitfähigkeit der Randzustände bei niedrigen Frequenzen zu machen. Schließlich nutzen wir in Kap. 7 die Methode der Bosonisierung sowie der Renormierungsgruppe um Luttinger-Flüssigkeitseffekte mit einzubinden.

9.3. Ergebnisse

In Kap. 5 berechneten wir die Leitfähigkeit von helikalen Fermionen in einem unendlich ausgedehnten Quantendraht bei hohen Frequenzen. Zunächst vernachlässigten wir Luttinger-Flüssigkeitseffekte, was uns in die Lage versetzte die Elektronen durch eine quasiklassische Boltzmann-Gleichung zu beschreiben. Streuterme wurden mit Fermis goldener Regel mit einbezogen. Wir führten eine perturbative Entwicklung sowohl in hohen Frequenzen als auch in Ordnungen der T-Matrix durch. Dies ist erlaubt unter der Annahme von schwacher Elektron-Elektron-Wechselwirkung und schwacher Störstellenstreuung.

In Abwesenheit jeglicher Streumechanismen ist die Leitfähigkeit: $\sigma^{(0)} = 2e^2/h(-i\omega)$, d.h. der entsprechende Leitwert ist exakt quantisiert. In reinen Systemen gibt es Korrekturen zur Leitfähigkeit aufgrund von g_5 Wechselwirkungsprozessen. Diese führen zu einer Temperaturabhängigkeit von T^5 bei hohen Temperaturen (Gl. 5.54), sind jedoch thermisch angeregt bei niedrigen Temperaturen (Gl. 5.51). Dies ist durch einfache Phasenraumargumente verständlich und wird in Kap. 5.2 diskutiert. Wie erwartet beeinflusst Störstellenstreuung die Transporteigenschaften aufgrund des topologischen Schutzes nicht. Die führenden Korrekturen in zweiter Ordnung der T-Matrix stammen von kombinierten Prozessen von Wechselwirkung und Störstellenstreuung. Die Temperaturabhängigkeit aufgrund dieser Prozesse ist T^4. Die Ergebnisse können mit denen für kurze Quantendrähte verglichen werden, indem wir die formale Substitution $(-i\omega) \to L_{\text{eff}}^{-1}$ durchführen. Dies erlaubt es uns die Leitfähigkeit bei hohen Frequenzen in einem langen Draht mit dem Gleichstromleitwert eines kurzen Drahtes zu vergleichen. Wir stellen fest das unsere Resultate mit denen früherer Veröffentlichungen [11] übereinstimmen.

Komplementär hierzu nutzten wir eine „operator product expansion“ um einen effektiven Hamiltonoperator herzuleiten, welcher die kombinierten Streuprozesse aus Störstellenstreuung und Wechselwirkung beinhaltet. Die erhaltenen Terme stimmen mit denen aus Ref. [12] überein. Nun berechneten wir die führenden Korrekturen zur Leitfähigkeit mit demselben quasiklassischen Formalismus wie zuvor. Wir kommen zu dem Ergebniss, dass bei niedrigen Temperaturen inelastische Streuprozesse am relevantesten sind und zu Korrekturen proportional T^4 führen:

$$\sigma_{\text{AC}}^{(1)} = -103.902 \frac{32}{\pi^4} \frac{e^2}{(-i\omega)^2 h} n_{\text{imp}} \left(\frac{k_F V_0 U_0}{k_0^2} \right)^2 T^4. \tag{9.1}$$

Obwohl dieses Vorgehen auf die gleiche Temperaturabhängigkeit der Leitfähigkeit führt, finden wir eine unterschiedliche Abhängigkeit von der Störstellenstärke U_0 und dem Grenzimpuls k_0. Diese Abhängigkeit ist nicht verstanden und muss noch weiter untersucht werden.

Kap. 6 beschäftigte sich mit dem Transportverhalten bei niedrigen Frequenzen. Wir untersuchten die Effekte von g_5-, inelastischen- und Zweiteilchen-Streuprozessen in unterschiedlichen Temperaturbereichen. Die Kombination von analytischen Manipulationen und numerischen Berechnungen erlaubte es uns die Leitfähigkeit für die relevantesten Prozesse exakt zu bestimmen. Wiederum stellen sich inelastische Prozesse als primärer Streumechanismus heraus und wir finden die Leitfähigkeit:

$$\sigma_{\mathrm{DC}} = \frac{2e^2}{h} 2.3 \times 10^{-2} \frac{\pi^4}{4k_F^2 n_{\mathrm{imp}}} \left(\frac{k_0^2}{U_0 V_0}\right)^2 T^{-4}. \tag{9.2}$$

Zudem zeigen wir, dass selbst in reinen Proben die Gleichstromleitfähigkeit nicht unendlich groß ist, da g_5 Prozesse zu einer exponentiell hohen aber endlichen Leitfähigkeit führen (siehe Gl. (6.51).

Schließlich sind wir in der Lage Gleich-und Wechselstromleitfähigkeiten zu vergleichen. Wir kommen zu dem Ergebniss, dass die Leitfähigkeit aufgrund unterschiedlicher numerischer Vorfaktoren in den meisten Fällen leicht von dem Drude Gesetz, Gl. (5.29), abweicht. g_5 Prozesse zeigen ein besonders interresantes Verhalten bei tiefen Temperaturen aufgrund des speziellen Zustands bei Impuls Null. Dies wird näher in Kap. 6.2.2 erläutert. Alle Ergebnisse und der Vergleich zwischen Gleich-und Wechselstromleitfähigkeit können der Tablle 6.1 entnommen werden.

In Kap. 7.1 bosonisieren wir das effektive Modell und führen einen Renormierungsgruppenschritt durch um die renormierten Kopplungskonstanten der Theorie zu erhalten. Diese verändern die Potenz der Temperaturabhängigkeit der Leitfähigkeit. Wir beschränken uns auf schwache Störstellenstreung, $T\tau \gg 1$ und identifizieren drei unterschiedliche Regime für Transporteigenschaften. Für den inelastischen Streuprozess finden wir folgende Ergebnisse in den unterschiedlichen Bereichen:

- (i) Hohe Frequenzen im Bereich $T \gg \omega \gg \tau^{-1}$

$$\sigma_{\mathrm{1P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 103.902 \frac{32}{\pi^4} \frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} k_F^2 (V_0 U_0)^2 \left(\frac{T}{k_0}\right)^{2K+2}. \tag{9.3}$$

- (ii) Hohe Frequenzen im Bereich $\omega \gg T \gg \tau^{-1}$

$$\sigma_{\mathrm{1P}}^{(\mathrm{AC})} = \frac{2e^2}{(-i\omega)h} - 103.902 \frac{32}{\pi^4} \frac{e^2}{(-i\omega)^2 h} n_{\mathrm{imp}} k_F^2 (V_0 U_0)^2 \left(\frac{T}{k_0}\right)^4 \left(\frac{\omega}{k_0}\right)^{2K-2}. \tag{9.4}$$

- (iii) Niedrige Frequenzen im Bereich $T \gg \tau^{-1} \gg \omega$

$$\sigma_{\mathrm{1P}}^{(\mathrm{DC})} = 2.3 \times 10^{-2} \frac{\pi^4}{n_{\mathrm{imp}}} \frac{1}{4k_F^2 U_0^2 V_0^2} \left(\frac{T}{k_0}\right)^{-2K-2}. \tag{9.5}$$

Dabei ist K der Luttingerflüssigkeitsparameter. Schließlich sind wir in der Lage Parameterregime zu identifizieren in denen Streuprozesse relevant (im Sinne der Renormierungsgruppe) werden könnten.

9.4. Ausblick

Die derzeitige experimentelle Situation (siehe Ref. [6]) legt nahe, dass der Leitwert für lange Proben um eine Größenordnung kleiner ist als für kurze. Zudem gibt es keine Anzeichen von Temperaturabhängigkeit der Leitfähigkeit. Dies ist in klarem Widerspruch zu der entwickelten Theorie. Alle Streumechanismen führen zu deutlichen Temperaturabhängigkeiten für Transporteigenschaften. Die einzige Ausnahme sind g_5 Wechselwirkungen, welche in gewissen Bereichen der Systemlänge temperaturunabhängig sind. Dieses Regime muss genauer untersucht werden. Insbesondere könnte es sich als entscheidend erweisen realistische Randbedingungen anzunehmen um einen makroskopisch langen, aber endlichen, Quantendraht zu simulieren. Ein anderer Ansatz ist die Untersuchung von schwachen Lokalisierungseffekten.

In gewöhnlichen eindimensionalen Systemen wird Lokalisierung nahezu vollständig durch die Theorie, die in Ref. [43, 44, 45] entwickelt wurde, beschrieben. Im Regime hoher Temperaturen ist die wechselwirkungsinduzierte Dephasierung groß. Betrachtet man jedoch die Situation, dass die Dephasierungsdauer τ_ϕ viel kleiner als die Transportstreuzeit τ_{tr} ist, so findet man nichtsdestotrotz eine negative Korrektur zum Leitwert durch schwache Lokalisierung. Verantwortlich dafür sind konstruktive Interferenzeffekte kohärenter Elektronenpfade. Um kohärent zu sein, müssen diese Pfade von Störstellenkonfigurationen stammen, die anomal nahe beieinander liegen. Die geringe Wahrscheinlichkeit derartiger Konfigurationen spiegelt sich im parametrisch kleinen Vorfaktor $(\tau_\phi/\tau_{\mathrm{tr}})^2$ der schwachen Lokalisierungskorrektur wieder. Der verwendete Formalismus zur Beschreibung der Problemstellung basiert auf einer perturbativen Behandlung von Störstellen im Rahmen der funktionalen Bosonisierung. Kürzlich wurde diese Methode erfolgreich zur Beschreibung schwacher Lokalisierung in zweisprossigen Leitermodellen verwendet [46], was ihre weitreichenden Anwendungsmöglichkeiten unter Beweis stellt.

Der fundamentale Unterschied zwischen diesem System und dem HLL ist, das im Letzteren keine elastische Streuung durch Störstellen existiert. Somit vermuten wir, wenigstens phänomenologisch, dass die Störstellenstreuung selbst zur Dephasierung beiträgt. Unsere bisherige Untersuchung der inelastischen Streuprozesse durch die quantenkinetische Gleichung hat uns wertvolle Einblicke in die zugrundeliegende Physik dieser Vorgänge gewährt. Somit wird es möglich sein den bestehenden Formalismus zur Behandlung schwacher Lokalisierung in HLL anzupassen. Wir erwarten dabei, dass die Korrektur durch schwache Lokalisierung bestehen bleibt, der Vorfaktor jedoch verschieden sein wird. Geht man überdies über das Regime der schwachen Lokalisierung hinaus, stellt sich zunächst die Frage, ob das Regime $\tau_\phi/\tau_{\mathrm{tr}} \sim 1$ überhaupt je erreicht werden kann. In bisherigen Modellen für Quantendrähte waren τ_ϕ und τ_{tr} unabhängig voneinander einstellbar: während τ_ϕ eine Funktion der Störstellenstärke ist, hängt τ_{tr} von der Wechselwirkungsstärke ab. Im Fall des HLL ist die Situation weit komplexer: es gibt keine elastische, sondern nur inelastische Streuung durch Störstellen und somit wird τ_{tr} von der Stärke der Störstellenstreuung abhängig, während τ_ϕ vorraussichtlich eine Abhängigkeit von der Wechselwirkungsstärke erhält. Sobald diese Abhängigkeiten verstanden sind, wird man in der Lage sein die Frage nach der Existenz eines stark lokalisierten Regimes (das bei $\tau_\phi/\tau_{\mathrm{tr}} \gtrsim 1$ auftritt) zu beantworten.
Zusammenfassend sollte untersucht werden, ob der topologische Schutz auch in Anwesenheit von Wechselwirkung und Störstellen bestehen bleibt oder gebrochen wird.

Die Untersuchung topologischer Isolatoren bleibt also weiterhin ein aufregendes Forschungsgebiet der Theorie kondensierter Materie, welches viele interessante Probleme für Theoretiker aufwirft.

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Bibliography

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).

[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).

[3] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).

[4] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).

[5] M. König *et al.*, Science **318**, 766 (2007).

[6] G. M. Gusev *et al.*, Phys. Rev. B **84**, 121302 (2011).

[7] C. Wu, B. A. Bernevig, and S.-C. Zhang, Phys. Rev. Lett. **96**, 106401 (2006).

[8] C. Xu and J. E. Moore, Phys. Rev. B **73**, 045322 (2006).

[9] F. m. c. Crépin, J. C. Budich, F. Dolcini, P. Recher, and B. Trauzettel, Phys. Rev. B **86**, 121106 (2012).

[10] J. Maciejko *et al.*, Phys. Rev. Lett. **102**, 256803 (2009).

[11] T. L. Schmidt, S. Rachel, F. von Oppen, and L. I. Glazman, Phys. Rev. Lett. **108**, 156402 (2012).

[12] N. Lezmy, Y. Oreg, and M. Berkooz, ArXiv e-prints (2012), 1201.6197.

[13] D. Hsieh *et al.*, Nature **452**, 970 (2008).

[14] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).

[15] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).

[16] D. Sénéchal, arXiv:cond-mat/9908262.

[17] A. Kamenev, *Field theory of non-equilibrium systems* (Cambridge Univ. Press, 2011).

[18] A. D. Mirlin, F. Evers, I. V. Gornyi, and P. M. Ostrovsky, International Journal of Modern Physics B **24**, 1577 (2010).

[19] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, arXiv:cond-mat.str-el/0905.2029.

[20] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New Journal of Physics **12**, 065010 (2010).

[21] J. J. Sakurai, *Modern quantum mechanics* (Addison-Wesley, 1994).

[22] R. Winkler, *Spin-orbit coupling effects in two-dimensional electron and hole systems : with 26 tables* (Springer, 2003).

[23] J. Fabian, A. Matos-Abiague, C. Ertler, P. Stano, and I. Žutić, Acta Physica Slovaca **57**, 565 (2007).

[24] B. Zhou, H.-Z. Lu, R.-L. Chu, S.-Q. Shen, and Q. Niu, Phys. Rev. Lett. **101**, 246807 (2008).

[25] T. Giamarchi, *Quantum physics in one dimension* (Clarendon Press, 2006).

[26] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and strongly correlated systems* (Cambridge University Press, 2004).

[27] A. Altland and B. D. Simons, *Condensed matter field theory* (Cambridge Univ. Press, 2012).

[28] J. L. Cardy, *Scaling and renormalization in statistical physics* (Cambridge Univ. Press, 2003).

[29] P. Ginsparg, arXiv:hep-th/9108028.

[30] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory* (Springer, 1997).

[31] L. Fritz, J. Schmalian, M. Müller, and S. Sachdev, Phys. Rev. B **78**, 085416 (2008).

[32] J. M. Kosterlitz and D. J. Thouless, Journal of Physics C: Solid State Physics **6**, 1181 (1973).

[33] H. Bruus and K. Flensberg, *Many-body quantum theory in condensed matter physics : an introduction* (Oxford Univ. Press, 2005).

[34] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, arXiv:cond-mat/0608590.

[35] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[36] N. Mott and W. Twose, Advanc. Phys. **10**, 107 (1961).

[37] K. Efetov, *Supersymmetry in disorder and chaos* (Cambridge Univ. Pr., 1997).

[38] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **46** (1992).

[39] J. Rammer, *Quantum field theory of non-equilibrium states* (Cambridge Univ. Press, 2007).

[40] E. M. Lifsic and L. P. Pitaevskij, *Lehrbuch der theoretischen Physik 10: Physikalische Kinetik* (Akad.-Verl., 1990).

[41] T. Giamarchi, Phys. Rev. B **46**, 342 (1992).

[42] A. Rosch and N. Andrei, Phys. Rev. Lett. **85**, 1092 (2000).

[43] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. **95**, 046404 (2005).

[44] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. B **75**, 085421 (2007).

[45] A. G. Yashenkin, I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. B **78**, 205407 (2008).

[46] M. P. Schneider, S. T. Carr, I. V. Gornyi, and A. D. Mirlin, Phys. Rev. B **86**, 155141 (2012).

Appendix

A. Derivation of the g-ology in the HLL

Here we derive a equivalent representation of the model describing helical edge states. The interaction hamiltonian we derived in Sec. 2.2.4 is:

$$\hat{H}_{\text{int}} = \frac{1}{L}\sum_{kqp}\sum_{\eta_1\eta_2\eta_3\eta_4} V(q)\left[B_k^\dagger B_{k-q}\right]_{\eta_1,\eta_2}\left[B_p^\dagger B_{p+q}\right]_{\eta_3,\eta_4}\hat{\psi}^\dagger_{k,\eta_1}\hat{\psi}_{k-q,\eta_2}\hat{\psi}^\dagger_{p,\eta_3}\hat{\psi}_{p+q,\eta_4}, \quad (10.1)$$

where the matrix elements are given by:

$$\left[B_k^\dagger B_p\right]_{\eta,\eta'} = \delta_{\eta,\eta'} + \eta\,\delta_{\bar{\eta},\eta'}(k^2-p^2). \quad (10.2)$$

We combine the two equations and keep only the terms containing one chirality factor which we will call g_5 terms. Let us look at these g_5 terms specifically and rewrite them in two ways,

$$\begin{aligned}\hat{H}_5 =& \frac{V_0}{L}\sum_{kpq}\sum_{\eta_1\eta_2\eta_3\eta_4}\left[\eta_1\delta_{\bar{\eta}_1\eta_2}\delta_{\eta_3\eta_4}(k^2-(k-q)^2)+\eta_3\delta_{\bar{\eta}_3\eta_4}\delta_{\eta_1\eta_2}(p^2-(p-q)^2)\right] \\ &\hat{\psi}^\dagger_{k,\eta_1}\hat{\psi}^\dagger_{p,\eta_3}\hat{\psi}_{p+q,\eta_4}\hat{\psi}_{k-q,\eta_2} \quad (10.3)\\ =& \frac{V_0}{L}\sum_{kpq}\sum_{\eta_1\eta_3}\Big[\eta_1(k^2-(k-q)^2)\hat{\psi}^\dagger_{k,\eta_1}\hat{\psi}^\dagger_{p,\eta_3}\hat{\psi}_{p+q,\eta_3}\hat{\psi}_{k-q,\bar{\eta}_1} \\ &+ \underbrace{\eta_3(p^2-(p+q)^2)\hat{\psi}^\dagger_{k,\eta_1}\hat{\psi}^\dagger_{p,\eta_3}\hat{\psi}_{p+q,\bar{\eta}_3}\hat{\psi}_{k-q,\eta_1}}_{\eta_1\leftrightarrow\eta_3:\quad \eta_1(p^2-(p+q)^2)\hat{\psi}^\dagger_{k,\eta_3}\hat{\psi}^\dagger_{p,\eta_1}\hat{\psi}_{p+q,\bar{\eta}_1}\hat{\psi}_{k-q,\eta_3}}\Big]. \quad (10.4)\end{aligned}$$

Now we decompose the sum over η_3 into parts with $\eta_3=\eta_1$ and $\eta_3=\bar{\eta}_1$ to get

$$\begin{aligned}\hat{H}_5 =& \frac{V_0}{L}\sum_{kpq}\sum_{\eta}\eta\Big[(k^2-(k-q)^2)\{\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\eta}\hat{\psi}_{k-q,\bar{\eta}}+\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\} \quad (10.5)\\ &+(p^2-(p+q)^2)\{\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\eta}+\hat{\psi}^\dagger_{k,\bar{\eta}}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\}\Big]. \quad (10.6)\end{aligned}$$

There are now two possibilities to combine these terms:

i) With the substitution $(k\leftrightarrow p)$, $q\to -q$ in line (10.6) the two lines become identical and add up. This leads to the g_5 terms in Eq. (2.37).

ii) The next combination is more tricky. Consider the second term in line (10.6). With the sequence of substitutions $(k\leftrightarrow p)$, $q\to q-k+p$ it takes a similar form to line (10.5) and they add to form:

$$((p+q)^2-(k-q)^2)\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}. \quad (10.7)$$

Under the sequence $p \to p-q$, $k \to k+q$, $p \leftrightarrow k$, $q \to -q$ this combination takes the form:

$$(k^2-p^2)\hat{\psi}^\dagger_{k+q,\bar{\eta}}\hat{\psi}^\dagger_{p-q,\eta}\hat{\psi}_{p,\bar{\eta}}\hat{\psi}_{k,\bar{\eta}}. \tag{10.8}$$

Now look at the first term in line (10.6) and perform the shifts $q \to q-p+k$ and $q \to -q$ to get:

$$(p^2-(k-q)^2)\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{k-q,\bar{\eta}}\hat{\psi}_{p+q,\eta}, \tag{10.9}$$

which can be combined with the corresponding term in line (10.5) to obtain:

$$(k^2-p^2)\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\eta}\hat{\psi}_{k-q,\bar{\eta}}. \tag{10.10}$$

After renaming $k \leftrightarrow p$ and reordering the operators we end up with:

$$-(k^2-p^2)\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p-q,\bar{\eta}}\hat{\psi}_{k+q,\eta}. \tag{10.11}$$

This is just the hermitian conjugate of (10.8) (the additional -1 comes from the fact that η and $\bar{\eta}$ are reversed and there is an additional prefactor of η).
Combining this form of the g_5 terms and the expression for the interaction Hamiltonian in Eq. (2.37) we can decompose the sum over η' in terms $\eta' = \eta$ and $\eta' = \bar{\eta}$ to obtain the following alternative form of the interaction hamiltonian:

$$\begin{aligned}\hat{H}_{\text{int}} = \frac{V_0}{L}\sum_{k,p,q,\eta} &\hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\eta,p}\hat{\psi}_{\eta,p+q}\hat{\psi}_{\eta,k-q} + \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\bar{\eta},p}\hat{\psi}_{\bar{\eta},p+q}\hat{\psi}_{\eta,k-q} \\ &- \frac{k^2-p^2}{k_0^2}\eta\left(\hat{\psi}^\dagger_{\eta,k+q}\hat{\psi}^\dagger_{\bar{\eta},p-q}\hat{\psi}_{\eta,p}\hat{\psi}_{\eta,k} + h.c.\right) \\ &+ \frac{\left(k^2-(k-q)^2\right)\left(p^2-(p+q)^2\right)}{k_0^4}\Big(\hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\eta,p}\hat{\psi}_{\bar{\eta},p+q}\hat{\psi}_{\bar{\eta},k-q} \\ &+ \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\bar{\eta},p}\hat{\psi}_{\bar{\eta},k-q}\hat{\psi}_{\eta,p+q}\Big).\end{aligned} \tag{10.12}$$

B. Calculations for the perturbation theory in the first order of the T matrix

Here we calculate the corrections to AC conductivity due to the interaction hamiltonian (10.12) in first order of the T-matrix.

g_4 process

First we calculate the matrix elements:

$$A^{kpq\eta}_{121'2'} = \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\eta}\hat{\psi}_{k-q,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle \tag{10.13}$$

$$= \delta_{\eta,\eta_{1'}}\delta_{\eta,\eta_{2'}}\delta_{\eta,\eta_1}\delta_{\eta,\eta_2}\left(\delta_{k,k_{2'}}\delta_{p,k_{1'}} - (1' \leftrightarrow 2')\right)\left(\delta_{k-q,k_1}\delta_{p+q,k_2} - (1 \leftrightarrow 2)\right). \tag{10.14}$$

If we sum this expression over all internal degrees of freedom it vanishes i.e.

$$\frac{V_0}{L}\sum_{k,p,q,\eta} A^{kpq\eta}_{121'2'} = 0. \tag{10.15}$$

Therefore there are no matrix elements for g_4 processes. Indeed transformed to real space the corresponding operator is local $(\hat{\psi}^\dagger_\eta\hat{\psi}^\dagger_\eta\hat{\psi}_\eta\hat{\psi}_\eta)(x)$ and therefore vanishes due to fermionic anticommutation relations.

g_2 process

The matrix elements read:

$$A^{kpq\eta}_{121'2'} = \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle \tag{10.16}$$

$$= \left(\delta_{k,k_{2'}}\delta_{p,k_{1'}}\delta_{\eta,\eta_{2'}}\delta_{\bar{\eta}\eta_{1'}} - (1' \leftrightarrow 2')\right)\left(\delta_{k-q,k_1}\delta_{p+q,k_2}\delta_{\eta,\eta_1}\delta_{\bar{\eta},\eta_2} - (1 \leftrightarrow 2)\right). \tag{10.17}$$

If summed over internal momenta and chiralities we get:

$$\begin{aligned}\frac{V_0}{L}\sum_{k,p,q,\eta} A^{kpq\eta}_{121'2'} = \frac{V_0}{L}\sum_{\eta}[&\delta_{\eta,\eta_{2'}}\delta_{\eta,\eta_1}\delta_{\bar{\eta},\eta_{1'}}\delta_{\eta,\eta_2} - \delta_{\eta,\eta_{1'}}\delta_{\eta,\eta_1}\delta_{\bar{\eta},\eta_{2'}}\delta_{\eta,\eta_2}\\ &- \delta_{\eta,\eta_{2'}}\delta_{\eta,\eta_2}\delta_{\bar{\eta},\eta_{1'}}\delta_{\eta,\eta_1} + \delta_{\eta,\eta_{1'}}\delta_{\eta,\eta_2}\delta_{\bar{\eta},\eta_{2'}}\delta_{\eta,\eta_1}]\\ &\times \delta_{k_1+k_2,k_{1'}+k_{2'}}.\end{aligned} \tag{10.18}$$

By using the symmetry properties of C the collision integral yields:

$$I_{k_1,\eta_1} = -2\pi\sum_{k_2,k_{1'},k_{2'}} 4C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\eta_1)(k_{2'},\bar{\eta}_1)}\delta_{k_1+k_2,k_{1'}+k_{2'}}. \tag{10.19}$$

The contribution to the conductivity vanishes since $\sum_{k_1,\eta_1} I_{k_1,\eta_1} = 0$ as can be readily checked by using the the symmetries of C.

g_3 process

The matrix elements read:

$$A^{kpq\eta}_{121'2'} = \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle \tag{10.20}$$

$$= \delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\bar{\eta}}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\eta}\left(\delta_{k,k_{2'}}\delta_{p,k_{1'}} - (1' \leftrightarrow 2')\right)\left(\delta_{k-q,k_1}\delta_{p+q,k_2} - (1 \leftrightarrow 2)\right). \tag{10.21}$$

Now we include the rest of the operator to get:

$$\frac{V_0}{k_0^4 L}\sum_{k,p,q,\eta}(k^2-(k-q)^2)(p^2-(p+q)^2)A^{kpq\eta}_{121'2'} \tag{10.22}$$

$$= \frac{V_0}{Lk_0^4}2h(\{k\})\delta_{k_1+k_2,k_{1'}+k_{2'}}\sum_{\eta}\delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\bar{\eta}}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\eta}, \tag{10.23}$$

where we defined the function

$$h(k_1,k_2,k_{1'},k_{2'}) := k_{1'}^2k_2^2 + k_1^2k_{2'}^2 - k_{1'}^2k_1^2 - k_{2'}^2k_2^2. \tag{10.24}$$

For the collision integral one gets:

$$I_{k_1,\eta_1} = -2\pi\frac{4V_0^2}{k_0^8L^2}\sum_{k_2,k_{1'},k_{2'}}[h(\{k\})]^2\,C_{(k_1,\eta_1)(k_2,\eta_1),(k_{1'},\bar{\eta}_1)(k_{2'},\bar{\eta}_1)}\delta_{k_1+k_2,k_{1'}+k_{2'}}. \tag{10.25}$$

The combination of momentum and energy conservation leads to the constraint $k_1 = -k_2$ and thus the prefactor vanishes i.e. $h(\{k\}) = 0$.

g_1 process

First we calculate the matrix elements:

$$A^{kpq\eta}_{121'2'} = \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\hat{\psi}_{p+q,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle \tag{10.26}$$

$$= \left(\delta_{k,k_{2'}}\delta_{p,k_{1'}}\delta_{\eta,\eta_{2'}}\delta_{\bar{\eta},\eta_{1'}} - (1' \leftrightarrow 2')\right)\left(\delta_{k-q,k_2}\delta_{p+q,k_1}\delta_{\eta,\eta_2}\delta_{\bar{\eta},\eta_1} - (1 \leftrightarrow 2)\right). \tag{10.27}$$

Next we calculate the remaining sums to get:

$$\frac{V_0}{k_0^4 L}\sum_{k,p,q,\eta}(k^2-(k-q)^2)(p^2-(p+q)^2)A^{kpq\eta}_{121'2'} \tag{10.28}$$

$$\begin{aligned}=&\frac{V_0}{Lk_0^4}\delta_{k_1+k_2,k_{1'}+k_{2'}}\sum_{\eta}\Big[\{(k_{2'}^2-k_2^2)(k_{1'}^2-k_1^2)\delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\eta}\delta_{\eta_{1'},\bar{\eta}}\delta_{\eta_{2'},\eta}\\ &-(1'\leftrightarrow 2')\}-(1\leftrightarrow 2)\Big].\end{aligned} \tag{10.29}$$

Thus the collision integral takes the form:

$$\begin{aligned}I_{k_1,\eta_1}=&-2\pi\frac{4V_0^2}{k_0^8L^2}\sum_{k_2,k_{1'},k_{2'}}\Big[\{(k_{2'}^2-k_2^2)^2(k_{1'}^2-k_1^2)^2C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\eta_1)(k_{2'},\bar{\eta}_1)}\\ &-(1'\leftrightarrow 2')\}-(1\leftrightarrow 2)\Big]\delta_{k_1+k_2,k_{1'}+k_{2'}}.\end{aligned} \tag{10.30}$$

The combination of energy and momentum conservation demands $k_1 = k_{1'}$ in the first term. Therefore the prefactor vanishes and consequently all the terms vanish.

C. Calculations for the perturbation theory in the second order of the T matrix

First we derive effective initial and final states that incorporate impurity scattering. As an example we calculate them explicitely for forward scattering:

$$\widetilde{|12\rangle}_f := \hat{G}_0\hat{H}_{\mathrm{imp},f}|12\rangle = \frac{U_0}{L}\sum_{kk'\eta}\frac{1}{\epsilon_i-\hat{H}_0+i\eta}\hat{\psi}^\dagger_{k+k',\eta}\hat{\psi}_{k,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle. \tag{10.31}$$

First we use the algebra of fermionic operators to get:

$$\hat{\psi}_{k,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle = \left(-\hat{\psi}^\dagger_1\hat{\psi}_{k,\eta}\hat{\psi}^\dagger_2+\hat{\psi}^\dagger_2\delta_{\eta,\eta_1}\delta_{k,k_1}\right)|0\rangle \tag{10.32}$$

$$= \left(-\hat{\psi}^\dagger_1\delta_{\eta,\eta_2}\delta_{k,k_2}+\hat{\psi}^\dagger_2\delta_{\eta,\eta_1}\delta_{k,k_1}\right)|0\rangle. \tag{10.33}$$

With this (10.31) becomes:

$$\widetilde{|12\rangle}_f = \frac{U_0}{L}\sum_{k'}\hat{G}_0\left(-\hat{\psi}^\dagger_{k_2+k',\eta_2}\hat{\psi}^\dagger_{k_1,\eta_1}+\hat{\psi}^\dagger_{k_1+k',\eta_1}\hat{\psi}^\dagger_{k_2,\eta_2}\right)|0\rangle. \tag{10.34}$$

The initial energy is $\epsilon_i = \epsilon_1+\epsilon_2$, where $\epsilon_{k,\eta} = \eta k$ and the action of the Greens function operator is given by

$$\hat{G}_0|\big(k_2+k',\eta_2\big),(k_1,\eta_1)\rangle = \frac{1}{\epsilon_1+\epsilon_2-\epsilon_{2+k'}-\epsilon_1+i0} = \frac{1}{\eta_2\left(k_2-k_2-k'\right)+i0} \tag{10.35}$$

$$= -\frac{1}{\eta_2 k'+i0}, \tag{10.36}$$

$$\hat{G}_0|\big(k_1+k',\eta_1\big),(k_2,\eta_2)\rangle = -\frac{1}{\eta_1 k'+i0}. \tag{10.37}$$

Thus we arrive at the expression:

$$\widetilde{|12\rangle}_f = \frac{U_0}{L}\sum_{k'}\left[\frac{1}{k'\eta_2+i0}\hat{\psi}^\dagger_{k_2+k',\eta_2}\hat{\psi}^\dagger_1-(1\leftrightarrow 2)\right]|0\rangle. \tag{10.38}$$

Analogously we find

$$\widetilde{\langle 1'2'|}_f = \langle 0|\frac{U_0}{L}\sum_{k'}\left[\frac{1}{k'\eta_{2'}+i0}\hat{\psi}_{1'}\hat{\psi}_{k_{2'}-k',\eta_{2'}} - \left(1'\leftrightarrow 2'\right)\right], \tag{10.39}$$

and for backward scattering

$$\widetilde{|12\rangle}_b = \frac{U_0}{k_0^2 L}\sum_{k'}\left[(k_1-k')\hat{\psi}^\dagger_{k',\bar{\eta}_1}\hat{\psi}^\dagger_2 - (1\leftrightarrow 2)\right]|0\rangle, \tag{10.40}$$

$$\widetilde{\langle 1'2'|}_b = \langle 0|\frac{U_0}{k_0^2 L}\sum_{k'}\left[(k'-k_{2'})\hat{\psi}_{1'}\hat{\psi}_{k',\bar{\eta}_{2'}} - \left(1'\leftrightarrow 2'\right)\right]. \tag{10.41}$$

C.1. g_4 and g_2 processes combined with backscattering off the impurity

Let us start with g_4 and calculate the matrix elements

$$\begin{aligned}
A^{kpqk'\eta}_{1'2',12} \equiv& \langle 1'2'|\hat{H}_4|\widetilde{12}\rangle_b && (10.42)\\
=&(k_1-k')\langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\eta}\hat{\psi}_{k-q,\eta}\hat{\psi}^\dagger_{k'.\bar{\eta}_1}\hat{\psi}^\dagger_2|0\rangle - (1\leftrightarrow 2) && (10.43)\\
=&(k_1-k')\delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\eta}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\eta}(\delta_{p,k_{1'}}\delta_{k,k_{2'}} - (1'\leftrightarrow 2'))\\
&\times(\delta_{k-q,k'}\delta_{k_2,p+q} - \delta_{p+q,k'}\delta_{k-q,k_2}) - (1\leftrightarrow 2), && (10.44)
\end{aligned}$$

and

$$\begin{aligned}
B^{kpqk'\eta}_{1'2',12} \equiv& \widetilde{\langle 1'2'|}_b\hat{H}_4|12\rangle && (10.45)\\
=&(k'-k_{2'})\langle 0|\hat{\psi}_{1'}\hat{\psi}_{k',\bar{\eta}_{2'}}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\eta}\hat{\psi}_{k-q,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle - (1'\leftrightarrow 2') && (10.46)\\
=&(k'-k_{2'})\delta_{\eta_1,\eta}\delta_{\eta_2,\eta}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\bar{\eta}}(\delta_{k,k'}\delta_{p,k_{1'}} - \delta_{p,k'}\delta_{k_{1'},k})\\
&\times(\delta_{p+q,k_2}\delta_{k-q,k_1} - (1\leftrightarrow 2)) - (1'\leftrightarrow 2'). && (10.47)
\end{aligned}$$

After summing over internal momenta and chiralities we find:

$$\sum_{k,p,q}\sum_{k',\eta}A^{kpqk'\eta}_{1'2',12} = 0 = \sum_{k,p,q}\sum_{k',\eta}B^{kpqk'\eta}_{1'2',12}. \tag{10.48}$$

Therefore this process does not contribute to the conductivity. Next we study g_2 processes combined with backscattering from the impurity. The corresponding matrix elements are:

$$\begin{aligned}
A^{kpqk'\eta}_{1'2',12} \equiv& \langle 1'2'|\hat{H}_2|\widetilde{12}\rangle_b && (10.49)\\
=&(k_1-k')\langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\eta}\hat{\psi}^\dagger_{k',\bar{\eta}_1}\hat{\psi}^\dagger_2|0\rangle - (1\leftrightarrow 2) && (10.50)\\
=&(k_1-k')(\delta_{p,k_{1'}}\delta_{k,k_{2'}}\delta_{\eta_{1'},\bar{\eta}}\delta_{\eta_{2'},\eta} - (1'\leftrightarrow 2'))\\
&\times(\delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\bar{\eta}}\delta_{k-q,k'}\delta_{p+q,k_2} - \delta_{\eta_1,\eta}\delta_{\eta_2,\eta}\delta_{p+q,k'}\delta_{k-q,k_2}) - (1\leftrightarrow 2), && (10.51)
\end{aligned}$$

and

$$\begin{aligned}
B^{kpqk'\eta}_{1'2',12} \equiv& \widetilde{\langle 1'2'|}_b\hat{H}_2|12\rangle && (10.52)\\
=&(k'-k_{2'})\langle 0|\hat{\psi}_{1'}\hat{\psi}_{k',\bar{\eta}_{2'}}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle - (1'\leftrightarrow 2') && (10.53)\\
=&(k'-k_{2'})(\delta_{k,k'}\delta_{p,k_{1'}}\delta_{\eta_{1'},\bar{\eta}}\delta_{\eta_{2'},\bar{\eta}} - \delta_{k_{1'},k}\delta_{p,k'}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\eta})\\
&\times(\delta_{k_1,k-q}\delta_{p+q,k_2}\delta_{\eta_1,\eta}\delta_{\eta_2,\bar{\eta}} - (1\leftrightarrow 2)) - (1'\leftrightarrow 2'). && (10.54)
\end{aligned}$$

After summing over internal momenta and chiralities we find that these contributions vanish too,

$$\sum_{k,p,q}\sum_{k',\eta}A^{kpqk'\eta}_{1'2',12} = 0 = \sum_{k,p,q}\sum_{k',\eta}B^{kpqk'\eta}_{1'2',12}. \tag{10.55}$$

C.2. g_3 with backscattering

Again we calculate the matrix elements:

$$A^{kpqk'\eta}_{1'2',12} \equiv \langle 1'2'|\hat{H}_2\widetilde{|12\rangle}_b \tag{10.56}$$

$$= (k_1 - k')\langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\hat{\psi}^\dagger_{k',\bar{\eta}_1}\hat{\psi}^\dagger_2|0\rangle - (1 \leftrightarrow 2) \tag{10.57}$$

$$\begin{aligned} =&(k_1 - k')\delta_{\eta_1,\eta}\delta_{\eta_2,\bar{\eta}}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\eta}(\delta_{k,k_{2'}}\delta_{p,k_{1'}} - (1' \leftrightarrow 2')) \\ &\times (\delta_{k',k-q}\delta_{p+q,k_2} - \delta_{k_2,k-q}\delta_{p+q,k'}) - (1 \leftrightarrow 2)\,, \end{aligned} \tag{10.58}$$

and

$$B^{kpqk'\eta}_{1'2',12} \equiv \widetilde{\langle 1'2'|}_b\hat{H}_2|12\rangle \tag{10.59}$$

$$= (k' - k_{2'})\langle 0|\hat{\psi}_{1'}\hat{\psi}_{k',\bar{\eta}_{2'}}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\eta}\hat{\psi}_{p+q,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle - (1' \leftrightarrow 2') \tag{10.60}$$

$$\begin{aligned} =&(k' - k_{2'})\delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\bar{\eta}}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\bar{\eta}}(\delta_{k,k'}\delta_{p,k_{1'}} - \delta_{k,k_{1'}}\delta_{p,k'}) \\ &\times (\delta_{k_1,k-q}\delta_{k_2,p+q} - (1 \leftrightarrow 2)) - (1' \leftrightarrow 2'). \end{aligned} \tag{10.61}$$

After summing over internal momenta and chiralities we arrive after some calculations at:

$$\sum_{k,p,q}\sum_{k',\eta} A^{kpqk'\eta}_{1'2',12}(k^2 - (k-q)^2)(p^2 - (p+q)^2) \tag{10.62}$$

$$\begin{aligned} =&2\sum_\eta \delta_{\eta_1,\eta}\delta_{\eta_2,\bar{\eta}}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\eta}(k_1 + k_2 - k_{1'} - k_{2'})(k^2_{1'} - k^2_{2'}) \\ &\times (k_2^2 - (k_{1'} + k_{2'} - k_2)^2) - (1 \leftrightarrow 2), \end{aligned} \tag{10.63}$$

and

$$\sum_{k,p,q}\sum_{k',\eta} B^{kpqk'\eta}_{1'2',12}(k^2 - (k-q)^2)(p^2 - (p+q)^2) \tag{10.64}$$

$$\begin{aligned} =&2\sum_\eta \delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\bar{\eta}}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\bar{\eta}}(k_1 + k_2 - k_{1'} - k_{2'})(k_1^2 - k_2^2) \\ &\times ((k_1 + k_2 - k_{1'})^2 - k^2_{1'}) - (1' \leftrightarrow 2'). \end{aligned} \tag{10.65}$$

These matrix elements lead to the following expression for the term $\sum_1 \eta_1 I_1$:

$$\begin{aligned} \sum_1 \eta_1 I_1 = \left(\frac{V_0U_0}{k_0^4L^2}\right)^2 (-8\pi)N_{imp}\sum_{\{k\}}\eta_1(k_1 + k_2 - k_{1'} - k_{2'})^2\Big[&(k^2_{1'} - k^2_{2'})^2 \\ &\times \{(k_2^2 - (k_{1'} + k_{2'} - k_2)^2)^2 C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\eta_1)(k_{2'},\eta_1)} \\ &+ (k_1^2 - (k_{1'} + k_{2'} - k_1)^2)^2 C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\bar{\eta}_1)(k_{2'},\bar{\eta}_1)}\} \\ &+ (k_1^2 - k_2^2)^2\{((k_1 + k_2 - k_{1'})^2 - k^2_{1'})^2 C_{(k_1,\eta_1)(k_2,\eta_1),(k_{1'},\bar{\eta}_1)(k_{2'},\eta_1)} \\ &+ ((k_1 + k_2 - k_{2'})^2 - k^2_{2'})^2 C_{(k_1,\eta_1)(k_2,\eta_1),(k_{1'},\eta_1)(k_{2'},\bar{\eta}_1)}\}\Big]. \end{aligned} \tag{10.66}$$

We can combine the terms using the symmetry properties of $C_{12,1'2'}$ to get

$$\begin{aligned} \sum_1 \eta_1 I_1 =& -32\pi\left(\frac{V_0U_0}{k_0^4L^2}\right)^2 N_{imp}\sum_{\{k\}}(k_1^2 - k_2^2)^2(k_1 + k_2 - k_{1'} - k_{2'})^2 \\ &\times ((k_1 + k_2 - k_{1'})^2 - k^2_{1'})C_{(k_1,R)(k_2,R),(k_{1'},L)(k_{2'},R)} \end{aligned} \tag{10.67}$$

$$= 256\pi\left(\frac{V_0U_0}{k_0^4L^2}\right)^2 N_{imp}\left(\frac{L}{2\pi}\right)^4 \frac{eE}{(-i\omega)T}T^{13}g(\zeta), \tag{10.68}$$

where we defined the function

$$\begin{aligned} g(\zeta) = &\int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, (\tilde{x}_1^2 - \tilde{x}_2^2)^2 \tilde{x}_3^2 ((\tilde{x}_1 + \tilde{x}_2 - \tilde{x}_3)^2 - \tilde{x}_3^2)^2 \\ &\times n(x_1) n(x_2)(1 - n(-x_3))(1 - n(x_1 + x_2 + x_3)). \end{aligned} \tag{10.69}$$

Here $\tilde{x}_{1,2} = x_{1,2} + \zeta$, $\tilde{x}_3 = x_3 - \zeta$. If $\zeta \gg 1$ we expand the polynomial and take only the leading order in ζ to get:

$$\begin{aligned} g(\zeta) =& 256\zeta^8 \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, (x_1 - x_2)^2 n(x_1) n(x_2)(1 - n(-x_3)) \\ &\times (1 - n(x_1 + x_2 + x_3)) + \mathrm{o}(\zeta^7) \end{aligned} \tag{10.70}$$

$$\approx 256\zeta^8 \times 103.93 + \mathrm{o}(\zeta^7). \tag{10.71}$$

C.3. g_1 with backscattering

We calculate the matrix elements:

$$A^{kpqk'\eta}_{1'2',12} \equiv \langle 1'2'|\hat{H}_1|\widetilde{12}\rangle_b \tag{10.72}$$

$$= (k_1 - k') \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\hat{\psi}_{p+q,\eta}\hat{\psi}^\dagger_{k',\bar{\eta}_1}\hat{\psi}^\dagger_2|0\rangle - (1 \leftrightarrow 2) \tag{10.73}$$

$$\begin{aligned} =&(k_1 - k')(\delta_{\eta_1,\bar{\eta}}\delta_{\eta_2,\bar{\eta}}\delta_{k',p+q}\delta_{k-q,k_2} - \delta_{\eta_1,\eta}\delta_{\eta_2,\eta}\delta_{k',k-q}\delta_{p+q,k_2}) \\ &\times (\delta_{p,k_{1'}}\delta_{k,k_{2'}}\delta_{\eta_{1'},\bar{\eta}}\delta_{\eta_{2'},\eta} - \left(1' \leftrightarrow 2'\right)) - (1 \leftrightarrow 2)\,, \end{aligned} \tag{10.74}$$

and

$$B^{kpqk'\eta}_{1'2',12} \equiv \widetilde{\langle 1'2'|}_b \hat{H}_1 |12\rangle \tag{10.75}$$

$$= (k' - k_{2'}) \langle 0|\hat{\psi}_{1'}\hat{\psi}_{k',\bar{\eta}_{2'}}\hat{\psi}^\dagger_{k,\eta}\hat{\psi}^\dagger_{p,\bar{\eta}}\hat{\psi}_{k-q,\bar{\eta}}\hat{\psi}_{p+q,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle - (1' \leftrightarrow 2') \tag{10.76}$$

$$\begin{aligned} =&(k' - k_{2'})(\delta_{p,k_{1'}}\delta_{k,k'}\delta_{\eta_{1'},\bar{\eta}}\delta_{\eta_{2'},\bar{\eta}} - \delta_{p,k'}\delta_{k,k_{1'}}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\eta}) \\ &\times (\delta_{\eta_1,\eta}\delta_{\eta_2,\bar{\eta}}\delta_{k_1,p+q}\delta_{k-q,k_2} - (1 \leftrightarrow 2)) - (1' \leftrightarrow 2'). \end{aligned} \tag{10.77}$$

Summing over internal degrees of freedom we get after some algebra:

$$\sum_{k,p,q}\sum_{k',\eta} A^{kpqk'\eta}_{1'2',12}(k^2 - (k-q)^2)(p^2 - (p+q)^2) \tag{10.78}$$

$$\begin{aligned} =& 2\sum_\eta \delta_{\eta_1,\eta}\delta_{\eta_2,\eta}\delta_{\eta_{1'},\eta}\delta_{\eta_{2'},\bar{\eta}}(k_1 + k_2 - k_{1'} - k_{2'}) \\ &\times \Big[(k_2^2 - k_{2'}^2)(k_{1'} + k_{2'} - k_2)^2 + k_{1'}^2(k_1^2 - k_2^2) \\ &+ (k_{2'}^2 - k_1^2)(k_{1'} + k_{2'} - k_1)^2\Big] - (1' \leftrightarrow 2'), \end{aligned} \tag{10.79}$$

and

$$\sum_{k,p,q}\sum_{k',\eta} B^{kpqk'\eta}_{1'2',12}(k^2 - (k-q)^2)(p^2 - (p+q)^2) \tag{10.80}$$

$$\begin{aligned} =& 2\sum_\eta \delta_{\eta_1,\eta}\delta_{\eta_2,\bar{\eta}}\delta_{\eta_{1'},\bar{\eta}}\delta_{\eta_{2'},\bar{\eta}}(k_1 + k_2 - k_{1'} - k_{2'}) \\ &\times \Big[(k_1^2 - k_2^2)(k_1 + k_2 - k_{2'})^2 + k_2^2(k_{2'}^2 - k_{1'}^2) \\ &+ (k_{1'}^2 - k_1^2)(k_1 + k_2 - k_{1'})^2\Big] - (1 \leftrightarrow 2). \end{aligned} \tag{10.81}$$

Note that g_1 and g_3 processes do not mix when taking the absolute square since they have distinct chiral factors. Proceeding analogously as in the case of g_3 processes, we get for $\sum_1 \eta_1 I_1$ after some simplifications:

$$\sum_1 \eta_1 I_1 = 256\pi \left(\frac{V_0 U_0}{k_0^4 L^2}\right)^2 N_{imp} \left(\frac{L}{2\pi}\right)^4 \frac{eE}{(-i\omega)T} T^{13} g_1(\zeta), \tag{10.82}$$

where we defined the function

$$\begin{aligned} g_1(\zeta) = \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, \tilde{x}_3^2 [(2\tilde{x}_3 - \tilde{x}_1)^2 (\tilde{x}_2^2 - \tilde{x}_3^2) + (\tilde{x}_1 + \tilde{x}_2 - \tilde{x}_3)^2 \\ \times (\tilde{x}_1^2 - \tilde{x}_2^2) + (2\tilde{x}_3 - \tilde{x}_2)^2 (\tilde{x}_3^2 - \tilde{x}_1^2)]^2 n(x_1) n(x_2) \\ \times (1 - n(-x_3))(1 - n(x_1 + x_2 + x_3)). \end{aligned} \tag{10.83}$$

We expand the polynomial as before and get for $\zeta \gg 1$:

$$g_1(\zeta) \approx \, 144 \times 1757.97 \zeta^6 + \mathrm{o}(\zeta^5). \tag{10.84}$$

Therefore these processes give only subleading corrections compared with the g_3 processes discussed previously.

C.4. g_5 with forward scattering

First we decompose the g_5 hamiltonian as follows:

$$\hat{H}_5 = -\frac{V_0}{k_0^2 L} \sum_{k,p,q,\eta} (k^2 - p^2) \eta \left(\hat{\psi}^\dagger_{\eta,k+q} \hat{\psi}^\dagger_{\bar{\eta},p-q} \hat{\psi}_{\eta,p} \hat{\psi}_{\eta,k} + h.c. \right) \tag{10.85}$$

$$= \hat{H}_5^{(I)} + h.c.\,, \tag{10.86}$$

$$\hat{H}_5^{(I)} \equiv -\frac{V_0}{k_0^2 L} \sum_{k,p,q,\eta} (k^2 - p^2) \eta \hat{\psi}^\dagger_{\eta,k+q} \hat{\psi}^\dagger_{\bar{\eta},p-q} \hat{\psi}_{\eta,p} \hat{\psi}_{\eta,k}. \tag{10.87}$$

Now we calculate the required matrix elements:

$$A^{kpqk'\eta}_{1'2',12} \equiv \widetilde{\langle 1'2'|}_f \hat{H}_5^{(I)} |12\rangle \tag{10.88}$$

$$= \frac{1}{\eta_{2'} k' + i0} \langle 0 | \hat{\psi}_{1'} \hat{\psi}_{k_{2'}-k',\eta_{2'}} \hat{\psi}^\dagger_{k+q,\eta} \hat{\psi}^\dagger_{p-q,\bar{\eta}} \hat{\psi}_{p,\eta} \hat{\psi}_{k,\eta} \hat{\psi}^\dagger_1 \hat{\psi}^\dagger_2 - (1' \leftrightarrow 2') \tag{10.89}$$

$$\begin{aligned} = \delta_{\eta_1,\eta} \delta_{\eta_2,\eta} (\delta_{k_1,k} \delta_{k_2,p} - (1 \leftrightarrow 2)) \Big[\frac{1}{\eta_{2'} k' + i0} \\ \times (\delta_{k_{2'}-k',k+q} \delta_{p-q,k_{1'}} \delta_{\eta_{1'},\bar{\eta}} \delta_{\eta_{2'},\eta} - \delta_{k_{2'}-k',p-q} \delta_{k+q,k_{1'}} \delta_{\eta_{1'},\eta} \delta_{\eta_{2'},\bar{\eta}}) \\ - (1' \leftrightarrow 2') \Big], \end{aligned} \tag{10.90}$$

$$B^{kpqk'\eta}_{1'2',12} \equiv \langle 1'2' | \hat{H}_5^{(I)} \widetilde{|12\rangle}_f \tag{10.91}$$

$$= \frac{1}{\eta_2 k' + i0} \langle 0 | \hat{\psi}_{1'} \hat{\psi}_{2'} \hat{\psi}^\dagger_{k+q,\eta} \hat{\psi}^\dagger_{p-q,\bar{\eta}} \hat{\psi}_{p,\eta} \hat{\psi}_{k,\eta} \hat{\psi}^\dagger_{k_2+k',\eta_2} \hat{\psi}^\dagger_1 - (1 \leftrightarrow 2) \tag{10.92}$$

$$\begin{aligned} = \frac{1}{\eta_2 k' + i0} \delta_{\eta_1,\eta} \delta_{\eta_2,\eta} (\delta_{k_2+k',k} \delta_{k_1,p} - \delta_{k_1,k} \delta_{k_2+k',p}) \\ \times (\delta_{k_{2'},k+q} \delta_{p-q,k_{1'}} \delta_{\eta_{1'},\bar{\eta}} \delta_{\eta_{2'},\eta} - (1' \leftrightarrow 2')) - (1 \leftrightarrow 2). \end{aligned} \tag{10.93}$$

The other matrix elements can be obtained by noticing:

$$\langle 1'2' | (\hat{H}_5^{(I)})^\dagger \widetilde{|12\rangle}_f = \left(\widetilde{\langle 12|}_f \hat{H}_5^{(I)} |1'2'\rangle \right)^\dagger = A^{kpqk'\eta}_{12,1'2'}, \tag{10.94}$$

$$\widetilde{\langle 1'2'|}_f (\hat{H}_5^{(I)})^\dagger |12\rangle = \left(\langle 12| \hat{H}_5^{(I)} \widetilde{|1'2'\rangle}_f \right)^\dagger = B^{kpqk'\eta}_{12,1'2'}. \tag{10.95}$$

After summing over internal degrees of freedom we get:

$$\sum_{k,p,q}\sum_{k',\eta} A^{kpqk'\eta}_{1'2',12}\eta(k^2-p^2) = 0, \tag{10.96}$$

$$\begin{aligned}\sum_{k,p,q}\sum_{k',\eta} B^{kpqk'\eta}_{1'2',12}\eta(k^2-p^2) =& 2\sum_{\eta}\eta\delta_{\eta_1,\eta}\delta_{\eta_2,\eta}(\delta_{\eta_{1'},\bar{\eta}}\delta_{\eta_{2'},\eta}-(1'\leftrightarrow 2'))\\ &\times\left[\frac{(k_{1'}+k_{2'}-k_1)^2-k_1^2}{\eta(k_{1'}+k_{2'}-k_1-k_2)+i0}-(1\leftrightarrow 2)\right].\end{aligned} \tag{10.97}$$

With these matrix elements we calculate the collision integral:

$$\begin{aligned}\sum_1 \eta_1 I_1 =& -32\pi\left(\frac{U_0V_0}{k_0^2L^2}\right)^2 N_{\text{imp}}\sum_{\{k\}} C_{(k_1,R)(k_2,R),(k_{1'},L)(k_{2'},R)}\\ &\times\frac{\left[(k_{1'}+k_{2'}-k_1)^2-k_1^2-(k_{1'}+k_{2'}-k_2)^2-k_2^2\right]^2}{(k_{1'}+k_{2'}-k_1-k_2)^2+\epsilon^2},\end{aligned} \tag{10.98}$$

where we renamed the infinitesimal $i0\to i\epsilon$. The expression can be rewritten to:

$$\sum_1 \eta_1 I_1 = 64\pi\left(\frac{U_0V_0}{k_0^2L^2}\right)^2 N_{imp}(\frac{L}{2\pi})^4\frac{eE}{(-i\omega)}T^4 g_2(\zeta), \tag{10.99}$$

with the dimensionless factor:

$$\begin{aligned}g_2(\zeta) =& \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,\frac{\left[(2x_3+x_2)^2-x_1^2-(2x_3+x_1)^2+x_2^2\right]^2}{4x_3^2+\epsilon^2}\\ &\times n(x-\zeta)n(x_2-\zeta)(1-n(-x_3-\zeta))(1-n(x_1+x_2+x_3-\zeta))\end{aligned} \tag{10.100}$$

$$= \int \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\frac{\left[x^2-y^2\right]^2}{4(z-\zeta)^2+\epsilon^2}\frac{1}{1+e^{x-z}}\frac{1}{1+e^{y-z}}\frac{e^{-z}}{1+e^{-z}}\frac{e^{x+y-z}}{1+e^{x+y-z}}, \tag{10.101}$$

where we made the substitutions $x = x_1 + x_3$, $y = x_2 + x_3$ and then $z = x_3 + \zeta$. The resulting integral has to be evaluated numerically. Plotting $g_2(\zeta)$ over ζ with $\epsilon = 10^{-2}$ in Fig. C.1, we find that it decays faster than ζ^{-1}. Therefore the correction due to these processes does not contribute in leading order in ζ for $\zeta \gg 1$.

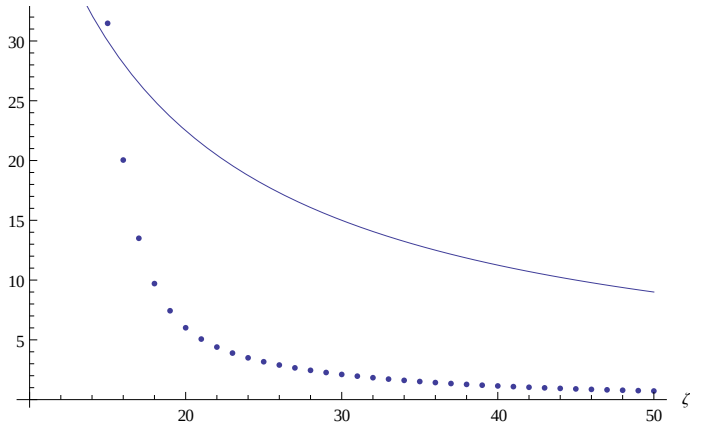

Figure C.1.: Dimensionless factor $g_2(\zeta)$ describing the temperature dependance of conductivity corrections due to g_5 interactions and impurity scattering (blue dots). As comparison a function of order ζ^{-1} is plotted as a blue line.

D. Calculation of the conductivity correction due to effective processes

Inelastic scattering

The generic Hamiltonian is:

$$\hat{H}_{1P} = -\frac{g_{1P}}{L^2} \sum_{k,p,q,q',\eta} k\, \hat{\psi}^\dagger_{q',\eta} \hat{\psi}^\dagger_{q,\bar{\eta}} \hat{\psi}_{p,\eta} \hat{\psi}_{k,\eta} + h.c.\,. \tag{10.102}$$

First we calculate the matrix elements:

$$\begin{aligned} A^{1'2'12}_{kpqq'\eta} &= \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{q',\eta}\hat{\psi}^\dagger_{q,\bar{\eta}}\hat{\psi}_{p,\eta}\hat{\psi}_{k,\eta}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle & (10.103)\\ &= \delta_{\eta_1\eta}\delta_{\eta_2\eta}\left(\delta_{k_1k}\delta_{k_2p} - (1\leftrightarrow 2)\right)\left(\delta_{\eta_{1'}\bar{\eta}}\delta_{\eta_{2'}\eta}\delta_{k_{1'},q}\delta_{k_{2'},q'} - (1'\leftrightarrow 2')\right), & (10.104)\\ B^{121'2'}_{kpqq'\eta} &= \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\eta,p}\hat{\psi}_{\bar{\eta},q}\hat{\psi}_{\eta,q'}\hat{\psi}^\dagger_1\hat{\psi}^\dagger_2|0\rangle & (10.105)\\ &= \delta_{\eta_{1'}\eta}\delta_{\eta_{2'}\eta}\left(\delta_{k_{1'}p}\delta_{k_{2'}k} - (1'\leftrightarrow 2')\right)\left(\delta_{\eta_1\eta}\delta_{\eta_2\bar{\eta}}\delta_{k_1,q'}\delta_{k_2,q} - (1\leftrightarrow 2)\right). & (10.106)\end{aligned}$$

With this we get:

$$\sum_{kpqq'\eta} A^{1'2'12}_{kpqq'\eta} k = \sum_\eta \delta_{\eta_1\eta}\delta_{\eta_2\eta}(k_1-k_2)(\delta_{\eta_{1'}\bar{\eta}}\delta_{\eta_{2'}\eta} - \delta_{\eta_{2'}\bar{\eta}}\delta_{\eta_{1'}\eta}), \tag{10.107}$$

$$\sum_{kpqq'\eta} B^{1'2'12}_{kpqq'\eta} k = \sum_\eta \delta_{\eta_{1'}\eta}\delta_{\eta_{2'}\eta}(k_{2'}-k_{1'})(\delta_{\eta_2\bar{\eta}}\delta_{\eta_1\eta} - \delta_{\eta_1\bar{\eta}}\delta_{\eta_2\eta}). \tag{10.108}$$

Therefore the collision integral reads:

$$\begin{aligned} I_{k_1,\eta_1} = -2\pi\frac{g^2_{1P}}{L^3} n_{\text{imp}} \sum_{k_2,k_{1'},k_{2'}} &[2(k_1-k_2)^2 C_{(k_1,\eta_1)(k_2,\eta_1),(k_{1'},\bar{\eta}_1)(k_{2'},\eta_1)} \\ &+ (k_{1'}-k_{2'})^2\left(C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\eta_1)(k_{2'},\eta_1)} + C_{(k_1,\eta_1)(k_2,\bar{\eta}_1),(k_{1'},\bar{\eta}_1)(k_{2'},\bar{\eta}_1)}\right)]. \end{aligned} \tag{10.109}$$

Next we use the symmetry properties of $C_{12,1'2'}$ to calculate the term,

$$\begin{aligned} \sum_1 \eta_1 I_1 =& -8\pi\frac{g^2_{1P}}{L^3} n_{\text{imp}} \sum_{\{k\}} (k_1-k_2)^2 C_{(k_1,R)(k_2,R),(k_{1'},L)(k_{2'},R)} & (10.110)\\ =& \frac{16\pi}{(2\pi)^4}\frac{eE}{(-i\omega)} n_{\text{imp}} g^2_{1P} T^4 f_{1P}(\zeta), & (10.111)\end{aligned}$$

where the resulting dimensionless integral is given by:

$$\begin{aligned} f_{1P}(\zeta) =& \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,(x_1-x_2)^2 n(x_1)n(x_2)(1-n(-x_3))(1-n(x_1+x_2+x_3)) & (10.112)\\ \approx& 103.9\,. & (10.113)\end{aligned}$$

Therefore the resulting conductivity correction is independant of the ratio between temperature and Fermi energy, ζ, and is given by:

$$\sigma^{(1)}_{1P} = -103.9\frac{2}{\pi^2}\frac{e^2}{h(-i\omega)} n_{imp} g^2_{1P} T^4. \tag{10.114}$$

Two-particle scattering

The effective hamiltonian for the two particle scattering process reads:

$$\hat{H}_{2P} = \frac{g_{2P}}{L^2} \sum_{k,p,q,q',\eta} kq\ \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\eta,p}\hat{\psi}_{\bar{\eta},q}\hat{\psi}_{\bar{\eta},q'}. \tag{10.115}$$

We start by calculating the matrix element:

$$A^{121'2'}_{kpqq'\eta} = \langle 0|\hat{\psi}_{1'}\hat{\psi}_{2'}\hat{\psi}^{\dagger}_{\eta,k}\hat{\psi}^{\dagger}_{\eta,p}\hat{\psi}_{\bar{\eta},q}\hat{\psi}_{\bar{\eta},q'}\hat{\psi}^{\dagger}_{1}\hat{\psi}^{\dagger}_{2}|0\rangle \tag{10.116}$$

$$= \delta_{1\bar{\eta}}\delta_{2\bar{\eta}}\delta_{1'\eta}\delta_{2'\eta}\left(\delta_{1q'}\delta_{2q} - (1 \leftrightarrow 2)\right)\left(\delta_{1'p}\delta_{2'k} - \left(1' \leftrightarrow 2'\right)\right). \tag{10.117}$$

After summation over internal degrees of freedom we arrive at:

$$\sum_{k,p,q,q',\eta} kqA^{121'2'}_{kpqq'\eta} = \sum_{\eta}\delta_{1\bar{\eta}}\delta_{2\bar{\eta}}\delta_{1'\eta}\delta_{2'\eta}(k_{2'} - k_{1'})(k_2 - k_1). \tag{10.118}$$

Thus the result for the collision integral reads:

$$I_1 = -2\pi\frac{g^2_{2P}}{L^3}n_{\text{imp}}\sum_{k_2,k_{1'},k_{2'}}(k_{2'} - k_{1'})^2(k_2 - k_1)^2 C_{(k_1,\eta_1)(k_2,\eta_1),(k_{1'},\bar{\eta}_1)(k_{2'},\eta_1)}. \tag{10.119}$$

Using the symmetry properties of C we get:

$$\sum_1 \eta_1 I_1 = -4\pi\frac{g^2_{2P}}{L^3}n_{\text{imp}}\sum_{\{k\}}(k_{2'} - k_{1'})^2(k_2 - k_1)^2 C_{(k_1,R)(k_2,R),(k_{1'},L)(k_{2'},L)} \tag{10.120}$$

$$= \frac{16\pi}{(2\pi)^4}n_{\text{imp}}g^2_{2P}\frac{eE}{(-i\omega)}T^6 f_{2P}(\zeta), \tag{10.121}$$

with

$$f_{2P}(\zeta) = \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,(2x_3 + x_1 + x_2)^2(x_1 - x_2)^2 n(x_1)n(x_2)(1 - n(-x_3)) \times (1 - n(x_1 + x_2 + x_3)) \approx 1757.97\,. \tag{10.122}$$

Therefore the conductivity correction is again independant of ζ and given by the following expression:

$$\sigma^{(1)} = -1757.97\frac{e^2}{h(-i\omega)^2}\frac{2}{\pi^2}n_{imp}g^2_{2P}T^6. \tag{10.123}$$

E. Exact form of the functions determining the integral equations for DC transport

We will often need the polylogarithm function which is defined as:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}, \tag{10.124}$$

for all complex orders of s and $|z| < 1$. It can also be analytically continued to the whole complex plane. The asymptotics of the polylogarithm function are given by:

$$\lim_{x\to\infty}\mathrm{Li}_n(-e^x) = -\frac{x^n}{\Gamma(n+1)} = -\frac{x^n}{n!}, \tag{10.125}$$

$$\lim_{|x|\to 0}\mathrm{Li}_n(x) = x, \tag{10.126}$$

$$\mathrm{Li}_n(-e^x) = -\eta(n) - \eta(n-1)x - \eta(n-2)x^2 + \mathcal{O}(x), \quad \text{for } |x| \ll 1. \tag{10.127}$$

The last limit is obtained by using the series representation (10.124) and expanding in small x. The Dirichlet eta function is given by:

$$\eta(n) = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^s}, \tag{10.128}$$

and can be calculated as:

$$\eta(n) = \begin{cases} \ln(2) & , \text{ for } n = 1 \\ (1 - 2^{1-n})\zeta(n) & , \text{ else.} \end{cases} \tag{10.129}$$

Here $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta function. Furthermore we will need the asymptotics of:

$$\mathcal{A}_{n,\zeta}(x) \equiv (n-1)! \left[\mathrm{Li}_n(-e^x) + (-1)^n \mathrm{Li}_n(-e^{-x}) - \mathrm{Li}_n(-e^{\zeta}) - (-1)^n \mathrm{Li}_n(-e^{-\zeta}) \right]. \tag{10.130}$$

For $\zeta \gg 1$ they are given by:

$$\mathcal{A}_{n,\zeta}(x) \approx -\frac{x^n - \zeta^n}{n}. \tag{10.131}$$

In the case where $|x| \gg 1$ this follows from the asymptotics Eq. (10.127). In the case that x is not large the terms with x can be neglected with respect to the terms concerning ζ and the approximation in Eq. (10.131) is still valid.

E.1. g_5 process

Calculation of $A_{\zeta,\pm}(x)$

The functions $A_{\zeta,\pm}(x)$ are defined in Sec. 6.2.1 as integrals over known functions $K_\zeta(x,y)$ and $G_\zeta(x,y)$. These integrals can be computed analytically and yield the following terms:

$$\int \mathrm{d}y \, K_\zeta(x,y) \equiv t_{K,\zeta}(x) p_{K,\zeta}(x), \tag{10.132}$$

$$\int \mathrm{d}y \, G_\zeta(x,y) \equiv t_{G,\zeta}(x) p_{G,\zeta}(x). \tag{10.133}$$

The thermal factors t and the polynomials p are defined as:

$$t_{K,\zeta}(x) = \frac{e^x + e^\zeta}{(-1+e^x)(1+e^\zeta)} \approx e^{-\zeta} \frac{e^x + e^\zeta}{(-1+e^x)}, \tag{10.134}$$

$$t_{G,\zeta}(x) = \frac{e^\zeta (e^x + e^\zeta)}{(1+e^\zeta)(-e^x + e^{2\zeta})} \approx \frac{(e^x + e^\zeta)}{(-e^x + e^{2\zeta})}, \tag{10.135}$$

$$p_{K,\zeta}(x) = 2x^5 - 4x^2 \mathcal{A}_{3,\zeta}(\zeta - x) + 2\mathcal{A}_{5,\zeta}(\zeta - x) \approx \frac{16}{15} x^5 + 2x^4 \zeta - 4x^2 \zeta^3 + 2x\zeta^4, \tag{10.136}$$

$$p_{G,\zeta}(x) = x^4(2\zeta - x) - 4x^3 \mathcal{A}_{2,\zeta}(x - \zeta) + 4x^2 \mathcal{A}_{3,\zeta}(x - \zeta) \approx -\frac{1}{3} x^2 (x - 2\zeta)^2. \tag{10.137}$$

Approximations are carried out in the regime $\zeta \gg 1$.

Calculation of $H_\zeta(y)$

The integrals defining $H_\zeta(y,z)$ can also be calculated exactly. Here we show the results and the approximations we made for $\zeta \gg 1$:

- $\int \mathrm{d}z \, H_\zeta(y,z) = t_{1,\zeta}(y) p_{1,\zeta}(y)$, with

$$t_{1,\zeta}(y) = \frac{\cosh(\frac{\zeta}{2})}{\sinh(\frac{y}{2} + \zeta)} \frac{1}{\cosh(\frac{y+\zeta}{2})}, \tag{10.138}$$

$$p_{1,\zeta}(y) = \frac{1}{2}(y + 2\zeta) y^4 + 2y^3 \mathcal{A}_{2,\zeta}(-y - \zeta) + 2y^2 \mathcal{A}_{4,\zeta}(x) \approx \frac{1}{6} y^2 (y + 2\zeta)^3. \tag{10.139}$$

- $\int \mathrm{d}y\, H_\zeta(y,z) = t_{2,\zeta}(y) p_{2,\zeta}(y)$, with

$$t_{2,\zeta}(y) = \frac{e^{\frac{z}{2}}}{-1+e^z} \frac{\cosh(\frac{\zeta}{2})}{\cosh(\frac{z+\zeta}{2})}, \tag{10.140}$$

$$p_{2,\zeta}(y) = 4z^2 \mathcal{A}_{3,\zeta}(\zeta - z) + 4z\mathcal{A}_{4,\zeta}(\zeta - z) + \mathcal{A}_{5,\zeta}(\zeta - z) \tag{10.141}$$

$$\approx \frac{8z^5}{15} - z^4\zeta + 2z^2\zeta^3 + z\zeta^4. \tag{10.142}$$

- $\int \mathrm{d}y\, H_\zeta(z-y,z) = t_{3,\zeta}(y) p_{3,\zeta}(y)$, with

$$t_{3,\zeta}(y) = \frac{e^y(1+e^\zeta)}{(-1+e^y)(e^y+e^\zeta)}, \tag{10.143}$$

$$p_{3,\zeta}(y) = y^5 - 2y^2\mathcal{A}_{3,\zeta}(\zeta - y) + \mathcal{A}_{5,\zeta}(\zeta - y) \approx \frac{8}{15}y^5 + y^4\zeta - 2y^2\zeta^3 - y\zeta^4. \tag{10.144}$$

- Using our approximate result from the integration over y we get an analytical approximation which holds for $\zeta >> 1$,

$$H_\zeta = \int \mathrm{d}y\mathrm{d}z\, H_\zeta(y,z) \tag{10.145}$$

$$\begin{aligned} =&\frac{1}{6}\cosh(\frac{\zeta}{2})\Big[\frac{16}{63}\frac{1}{\cosh(\frac{\zeta}{2})}\left(2\pi^6 - 945\left[\mathrm{Li}_6(-e^\zeta) + \mathrm{Li}_6(-e^{-\zeta})\right]\right) \\ &+ 192\zeta\frac{1}{\cosh(\frac{\zeta}{2})}\left[\mathrm{Li}_5(-e^\zeta) - \mathrm{Li}_5(-e^{-\zeta})\right] + 4\zeta^2(\pi^2+\zeta^2)^2\frac{\sinh(\frac{\zeta}{2})}{\sinh(\zeta)}\Big] \end{aligned} \tag{10.146}$$

$$\approx \frac{11}{90}\zeta^6. \tag{10.147}$$

The functions defined in Sec. 6.2.1 are now obtained as:

$$H_{\zeta,1}(y) = -t_{1,\zeta}(-y) p_{1,\zeta}(-y), \tag{10.148}$$

$$H_{\zeta,2}(y) = t_{2,\zeta}(-y) p_{2,\zeta}(-y), \tag{10.149}$$

$$H_{\zeta,3}(y) = t_{3,\zeta}(y) p_{3,\zeta}(y). \tag{10.150}$$

We will also need H_ζ for $\zeta \ll 1$. This result is obtained by expanding Eq. (10.145) in small ζ as $H_\zeta = H_0 + \partial_\zeta H_\zeta|_{\zeta=0}\,\zeta + \cdots$. The necessary integrals are calculated numerically and we obtain

$$H_\zeta = 480.7 + 161.3\,\zeta^2 + 12.6\,\zeta^4 + \mathcal{O}(\zeta^6). \tag{10.151}$$

All approximations have been checked numerically and are in very good agreement with the exact expressions.

E.2. Inelastic process

Calculation of $B_{1,\zeta}(x)$

We find:

$$\int \mathrm{d}z\, B_{1,\zeta}(x,y,z) = t_{B_1,\zeta}(x,y) p_{B_1,\zeta}(x,y), \tag{10.152}$$

where we defined the functions,

$$t_{B_1,\zeta}(x,y) = \frac{1+e^{x-\zeta}}{(1+e^{\zeta-y})(1-e^{x+y-2\zeta})}, \tag{10.153}$$

$$\begin{aligned} p_{B_1,\zeta}(x,y) =& 2(x^2+y^2)(2\zeta-x-y) - 4(x+y)\mathcal{A}_{2,\zeta}(x+y-\zeta) \\ &+ 4\mathcal{A}_{3,\zeta}(x+y-\zeta). \end{aligned} \tag{10.154}$$

If ζ were very large we could approximate the polylogarithm by its asymptotics and calculate the integral over y with the approximated function. The end result is

$$\begin{aligned} B_{1,\zeta}(x) =& \frac{4}{3}(x-\zeta)^2(\pi^2 - 3\mathcal{B}_{2,\zeta}(x-\zeta)) - 8(\zeta-x)\mathcal{B}_{3,\zeta}(x-\zeta) + \frac{8\pi^4}{45} \\ &- 8\mathcal{B}_{4,\zeta}(x-\zeta). \end{aligned} \tag{10.155}$$

Here we defined:

$$\mathcal{B}_{n,\zeta}(x) = Li_n(-e^{x-\zeta}) - (-1)^n Li_n(-e^{-x+\zeta}) \tag{10.156}$$

We note that $B_{1,\zeta}(x) = B_1(x-\zeta)$ and that its proportional to ζ^4. Thus a resonable ansatz would be

$$B_{1,\zeta}(x) = (x-\zeta)^4 + \pi^4 \tag{10.157}$$

where the π^4 comes from calculating $B_{1,\zeta}(\zeta)$ numerically. We checked numerically that this is a good approximation for all values of ζ.

Approximations for $M_{1,\zeta}(x,y,z)$

One integral can again be done analytically yielding:

$$\int \mathrm{d}z\, M_{1,\zeta}(x,y,z) = t_{M_1,\zeta}(x,y) p_{M_1,\zeta}(x,y). \tag{10.158}$$

The thermal factor and the polynomial are given by:

$$t_{M_1,\zeta}(x,y) = \frac{1+e^{x-\zeta}}{(1+e^{\zeta-y})(e^{x+y-2\zeta}-1)}, \tag{10.159}$$

$$p_{M_1,\zeta}(x,y) = 4xy(2\zeta-x-y) - 4(x+y)\mathcal{A}_{2,\zeta}(x+y-\zeta) + 4\mathcal{A}_{3,\zeta}(x+y-\zeta) \tag{10.160}$$

$$\approx \frac{2}{3}(x+y-2\zeta)(x^2-4xy+y^2+2(x+y)\zeta-2\zeta^2). \tag{10.161}$$

Under the assumption $x \approx \zeta$, motivated in Sec. 6.2.2 in the main text, we approximate $t_{B_1,\zeta}(x,y) \approx t_{B_1,\zeta}(\zeta,y)$. In the polynomial we use the asymptotics of polylogarithm functions Eq. (10.131) and furthermore set $(2\zeta-x-y) \approx (\zeta-y)$. This is necessary to keep the analytical properties of M_1 intact.[1]. Summarizing,

$$\int \mathrm{d}z\, M_{1,\zeta}(x,y,z) \approx \frac{2}{3}\frac{y-\zeta}{\sinh(y-\zeta)}(x^2-4xy+y^2+2(x+y)\zeta-2\zeta^2). \tag{10.162}$$

E.3. Two particle process

Approximations for $M_{2,\zeta}(x,y,z)$

First we calculate the kernel by evaluating the following integrals:

[1] Note that M_1 has a removable singularity at $y = 2\zeta - x$.

(A) Integral over y,

$$M_{A,\zeta}(x,z) \equiv \int \mathrm{d}y\, M_{2,\zeta}(x,y,z) = t_{A,\zeta}(x,z) p_{A,\zeta}(x,z). \tag{10.163}$$

Here the thermal factor and the polynomial are defined as:

$$t_{A,\zeta}(x,z) = \frac{e^z(e^x+e^\zeta)}{(-1+e^{x+z})(1+e^{z+\zeta})}, \tag{10.164}$$

$$\begin{aligned} p_{A,\zeta}(x,z) =& (x^4+4x^3z+4x^2z^2)(x+z) - 4(x^2z+2xz^2) \\ &\times \mathcal{A}_{2,\zeta}(-x-z+\zeta) + (-2x^2-4xz+4z^2)\mathcal{A}_{3,\zeta}(-x-z+\zeta) \\ &+ 4z\mathcal{A}_{4,\zeta}(-x-z+\zeta) + \mathcal{A}_{5,\zeta}(-x-z+\zeta). \end{aligned} \tag{10.165}$$

(B) Integral over z,

$$M_{B,\zeta}(x,y) \equiv \int \mathrm{d}z\, M_{2,\zeta}(x,y,z) = t_{B,\zeta}(x,z) p_{B,\zeta}(x,z). \tag{10.166}$$

Here we defined:

$$t_{B,\zeta}(x,z) = \frac{e^{y+\zeta}(e^x+e^\zeta)}{(e^y+e^\zeta)(e^{2\zeta}-e^{x+y})}, \tag{10.167}$$

$$\begin{aligned} p_{B,\zeta}(x,z) =& (x^4-2x^2y^2+y^4)(2\zeta-x-y) \\ &- 4(x^3-x^2y-xy^2+y^3)\mathcal{A}_{2,\zeta}(x+y-\zeta) \\ &+ (4x^2-8xy+4y^2)\mathcal{A}_{3,\zeta}(x+y-\zeta). \end{aligned} \tag{10.168}$$

Now we are able to write down the approximate expression for the integral kernel,

$$\int \mathrm{d}z\ (M_{2,\zeta}(x,y,z) + 2M_{2,\zeta}(x,z,-y)) = (M_{B,\zeta}(x,y) + 2M_{A,\zeta}(x,-y)). \tag{10.169}$$

Explicitely,

$$M_{B,\zeta}(x,y) \approx \frac{e^{y+\zeta}(e^x+e^\zeta)}{(e^y+e^\zeta)(e^{2\zeta}-e^{x+y})} \left[\frac{1}{3}(x-y)^2(2\zeta-x-y)^3\right], \tag{10.170}$$

$$\begin{aligned} M_{A,\zeta}(x,-y) \approx& \frac{e^y(e^x+e^\zeta)}{(e^{2\zeta}-e^{x+y})(e^y+e^\zeta)} \frac{1}{15}(x-y)\big[8x^4-47x^3y+93x^2y^2-47xy^3 \\ &+8y^4+15(x+y)(x^2-4xy+y^2)\zeta+90xy\zeta^2-30(x+y)\zeta^3+15\zeta^4\big]. \end{aligned} \tag{10.171}$$

Now we approximate the thermal factors by their value at $x=\zeta$ to get:

$$\frac{e^{y+\zeta}(e^x+e^\zeta)}{(e^y+e^\zeta)(e^{2\zeta}-e^{x+y})} \approx -\frac{1}{\sinh(y-\zeta)}, \tag{10.172}$$

$$\frac{e^{y}(e^x+e^\zeta)}{(e^{2\zeta}-e^{x+y})(e^y+e^\zeta)} \approx -\frac{1}{\sinh(y-\zeta)}. \tag{10.173}$$

This allows us to combine the polynomials. However, to keep the analytic structure of the functions correct we have to pull a factor $(\zeta-y)$ out of each poynomial. Consequently the form we will be using is:

$$\int \mathrm{d}y\, (M_{B,\zeta}(x,y) + 2M_{A,\zeta}(x,-y)) \frac{\psi_{\zeta,R}(y)}{D} \tag{10.174}$$

$$\approx -\int \mathrm{d}y \frac{(\zeta-y)}{\sin(y-\zeta)} \left(p_{I,\zeta}(x,y) + 2p_{II,\zeta}(x,y)\right) \frac{\psi_{\zeta,R}(y)}{D} \tag{10.175}$$

$$= \int \mathrm{d}y \frac{(y-\zeta)}{\sin(y-\zeta)} p_\zeta(x,y) \frac{\psi_{\zeta,R}(y)}{D}, \tag{10.176}$$

where

$$p_{I,\zeta}(x,y) = \frac{1}{3}(x-y)^2(x+y-2\zeta)^2, \tag{10.177}$$

$$\begin{aligned} p_{II,\zeta}(x,y) =& \frac{1}{15}\big[8x^4-47x^3y+93x^2y^2-47xy^3+8y^4+15(x+y)(x^2-4xy+y^2)\zeta \\ &+90xy\zeta^2-30(x+y)\zeta^3+15\zeta^4\big], \end{aligned} \tag{10.178}$$

$$p_\zeta(x,y) = p_I(x,y,\zeta) + p_{II}(x,y,\zeta). \tag{10.179}$$

Calculation of $B_{2,\zeta}(x)$

For B_2 we find the following approximation:

$$B_{2,\zeta}(x) = \int \mathrm{d}y\, M_{B,\zeta}(x,y) \tag{10.180}$$

$$\approx -\frac{1}{3}\int \mathrm{d}y\, \frac{e^{y+\zeta}(e^x+e^\zeta)}{(e^y+e^\zeta)(e^{2\zeta}-e^{x+y})}(x-y)^2(x+y-2\zeta)^3 \tag{10.181}$$

$$= -\frac{1}{3}\int \mathrm{d}\tilde{y}\, \frac{e^{-x+\tilde{y}+3\zeta}(e^x+e^\zeta)}{(e^{\tilde{y}-x+2\zeta}+e^\zeta)(e^{2\zeta}-e^{\tilde{y}+2\zeta})}(2(x-\zeta)-\tilde{y})^2\tilde{y}^3, \tag{10.182}$$

where we made the substitution $\tilde{y} = x+y-2\zeta$. Thus we get

$$B_{2,\zeta}(x) \approx 4(x-\zeta)^2 g_{0,\zeta}(x,\zeta) - 4(x-\zeta)g_{1,\zeta}(x) + g_{2,\zeta}(x), \tag{10.183}$$

where

$$g_{n,\zeta}(x) = -\frac{1}{3}\int \mathrm{d}\tilde{y}\, \frac{e^{-x+\tilde{y}+3\zeta}(e^x+e^\zeta)}{(e^{\tilde{y}-x+2\zeta}+e^\zeta)(e^{2\zeta}-e^{\tilde{y}+2\zeta})}\tilde{y}^{3+n}, \tag{10.184}$$

$$g_{0,\zeta}(x) = \frac{2\pi^4}{45} - 2\left[Li_4(-e^{x-\zeta}) + Li_4(-e^{-x+\zeta})\right], \tag{10.185}$$

$$g_{1,\zeta}(x) = -8\left[Li_5(-e^{x-\zeta}) - Li_5(-e^{-x+\zeta})\right], \tag{10.186}$$

$$g_{2,\zeta}(x) = \frac{16\pi^6}{189} - 40\left[Li_6(-e^{x-\zeta}) - Li_6(-e^{-x+\zeta})\right]. \tag{10.187}$$

We see that B_2 is a function of $(x-\zeta)$ only and goes as $(x-\zeta)^6$. Thus we approximate it as:

$$B_{2,\zeta}(x) = \frac{11}{90}(x-\zeta)^6 + \frac{\pi^6}{2}. \tag{10.188}$$

The approximation has be checked numerically and is in good agreement with the exact expression independant of the value of the parameter ζ.